The Convex Hull Prob
(Sorting Prob of CG)

Def $A \subseteq \mathbb{R}^d$ is convex if closed under convex combinations.

Def $\text{ConvexClosure}(A) = \text{CC}(A) =$ smallest convex set $\supseteq A$

2 Defs of Convex Hull

Def 1 $\text{CH}(A) = \bigcup \text{CC}(A)$

Def 2 $\text{CH}(A) = \text{CC}(A)$

We will use Def 1

A finite set

Thus in 2D $\text{CH}(A)$ is a simple closed polygon.

(say CCW)
We will use the following characterization:

Claim: \([a, b] \) is on \(CH(A)\) if and only if \(a \neq b\)

1) \(a, b \in A\)

2) \(\forall a' \in A \) either \(a'\) left of \([a, b]\)
   \[a' \in [a, b]\]

2D Convex Hull by divide-and-conquer

\[A = \{P_1, \ldots, P_n\} \]

Preprocess: sort \(A\) by \(x\)-coordinate

2D-CH \((A)\)

if \(|A| = 1\) return \(P_i\)

else \(CH_L = 2D-CH(P_1, \ldots, P_{\frac{n}{2}})\)

\(CH_R = 2D-CH(P_{\frac{n}{2}+1}, \ldots, P_n)\)

STITCH \((CH_L, CH_R)\)
**STITCH** \((L, R)\)

- **Lower bridge** \((L, R)\)
  - \(a = \text{right most}(L)\)
  - \(b = \text{left most}(R)\)

- \(*\) While \(a \text{ left}(a, b)\) set \(a \leftarrow a\)
- \(**\) While \(b \text{ left}(a, b)\) set \(b \leftarrow b\)

**Upper bridge** \((L, R) = \)

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**Correctness**

- \(*\) generates triangles \((a, b, a)\)
- \(**\) " \((a, b, b)\)

1) The \(\Delta\)'s are disjoint
   - They are ordered by their intersection with vertical line \(L\).
2) They are in \(\text{CC}(A)\).

Thus, termination:

At termination, \(a, \bar{a}, b, \bar{b}\) are all lift of \((a, b)\).
Since \((2, 2), (a, f), (b, b), (b, f)\) are on \(CH(L)\) \& \(CH(R)\) respectively.

Done

\[\text{Timing: Preprocess } O(n \log n) \text{ to sort} \]

\[\text{STITCH in } O(n)\]

\[T(n) = 2T(n/2) + cn\]

\[\therefore T(n) = O(n \log n)\]
Lower bounds

Sorting reducible to CH

Input: $x_1, \ldots, x_n$

$\text{CH}\left( (x_1, x_1^2), \ldots, (x_n, x_n^2) \right)$

The CH will be $x_i$'s in sorted order.

An important use for CH

$P_i, \ldots, P_n \in \mathbb{R}^2$

$\bar{P}_i = (P_x, P_y, P_x^2 + P_y^2)$

$\text{CH}(\bar{P}_1, \ldots, \bar{P}_n) = \text{Triangulated surface}$

Delaunay Triangulation
**Random Incremental CH**

**Procedure Random Incremental CH (P)**

1. Make $\Delta = (P_1, P_2, P_3)$ pick $C \in$ interior $\Delta$
2. Construct ray from $C$ to each $P_i$
3. Partition $P_i$ by edge of $\Delta$ they cross.
4. Randomly permute $P_1, \ldots, P_n$.

   For $i = 1$ to $n$
   
   let $e$ be edge crossed by ray $C \rightarrow P_i$
   
   BuildTent $(P, e)$

**Procedure BuildTent (P, e)**

1. Find edges of CH "visible" to $P$ by searching out from $e$
2. Replace visible edges with 2 new edges
3. Assign rays to the new edges.
Correctness?

Timing

$O(n)$ work other than BuildTree.

Consider steps 1 & 2 in BuildTree:

1) At most an edge

a) Charge rule for line-side tests
   
   2 not visible test we charge $p_i$
   
   each visible test we charge to the edge

   total $2n + 2n = 4n$ tests.

Consider step 3 in BuildTree:

Ray-costs

Backwards analysis

3 points to pick from say $p_i$

$\text{Cost}(p_i) = \begin{cases} 0 & \text{if } p_i \text{ not on hull} \\ \# \text{ray crossing to left & right} & 0, w. \end{cases}$
\(C_i = \text{cost}\)

\[E(C_i) = \frac{2(n-i)}{i-3}\]

\(C = \text{total cost}\)

\[E(C) = \sum_{i=4}^{n} E(C_i) \leq \sum_{i=4}^{n} \frac{2(n-i)}{i-3} \leq 2n \sum_{i=1}^{n} \frac{1}{i}\]