15-456/852 Computational Geometry, Fall 2017

Take Home Final (90 pts) Due: Dec 15
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<tr>
<th>Question</th>
<th>Points</th>
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<td>1</td>
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**Instructions:** Below please find 5 questions.

If you are taking the class 15-456 you should do any 3 question for full credit. If you are taking the class 15-852 then you should attempt any 4 questions for full credit. Solving extra questions will be extra credit.

For this final, you may cite any algorithm we did in class, or any algorithm described in the notes.

You may also cite homework problems. You may ask clarifying questions on Piazza, but do not expect substantial hints.

(30) 1. **Projection onto convex body contracts distances**

Let $C \subset \mathbb{R}^k$ be a convex body, and let $f_C : \mathbb{R}^k \to C$ be the function taking point $x$ to the closest point on $C$ to $x$. Show that

$$d(x, y) \geq d(f(x), f(y))$$

for all points $x, y \in \mathbb{R}^k$

(30) 2. **Separating Boxes with a Linear Separator**

Show that two axis aligned boxes that are disjoint can always be separated by an axis parallel hyperplane.

(Exercise 19.1, Har-Peled book).

(30) 3. **Separating 2-D points with an Axis-Aligned Ellipse**

An axis aligned ellipse is one whose two axes are parallel to the lines $x = 0$ and $y = 0$ respectively.
Given a set of red points $R$ and a set of green points $G$ in the 2D plane, give an algorithm to find an axis-aligned ellipse $E$ such that all points $R$ are on the outside of the ellipse, and all points $G$ are either on the boundary or on the inside of the ellipse – if such an ellipse exists. Your algorithm should run in expected linear time in the size of $R$ and $G$.

(40) 4. Brunn Minkowski Inequality, Slight Extension

For $A$ and $B$ be compact sets in $\mathbb{R}^n$, we have for any $\lambda \in [0, 1]$ that $\text{Vol}(\lambda A + (1 - \lambda)B) \geq \text{Vol}(A)^\lambda (\text{Vol}(B))^{1-\lambda}$. (Exercise 19.2, Har-Peled book).

(40) 5. Approximate Euclidean MSTs

Suppose you’re given a 4 well-separated pair decomposition $\{\{A_1, B_1\}, \{A_2, B_2\}, \ldots \{A_m, B_m\}\}$ of an $n$ point set in $\mathbb{R}^d$, for fixed dimension $d$. For each pair $\{A_i, B_i\}$ in the well separated pair, you’re given oracle access to a point $a \in A_i$ and $b \in B_i$ such that

$$d(a, b) \leq (1 + \epsilon)d(A_i, B_i)$$

Assume you are given the well-separated pair decomposition and the oracle, so no processing time is needed for those. Show that you can construct a spanning tree of your points in $O(n \log n)$ time, such that the sum of the edge weights of the spanning tree is within a $(1 + \epsilon)$ multiplicative factor of the sum of the edge weights of the Euclidean minimum spanning tree on $P$.

To avoid any possible confusion, a 4-WSPD if each pair $\{Q, R\}$ satisfies

$$\max(\text{diam}(Q), \text{diam}(R)) \leq \frac{1}{4}d(Q, R)$$