1. **Sampling from Spheres**

In this problem we will devise efficient algorithms for sampling uniformly from the spheres of common norms in high-dimensional spaces. The norms we will look at are:

- $L_1$, defined as $\|x\|_1 := \sum_i |x_i|$,
- $L_2$, defined as $\|x\|_2 := \sqrt{\sum_i x_i^2}$, and
- $L_\infty$, defined as $\|x\|_\infty := \max_i |x_i|$.

You may use the following theorem without proof:

**Theorem 0.1** Let $\|\cdot\|$ be one of the $L_1$, $L_2$, $L_\infty$ norms over $\mathbb{R}^d$. Let $X$ be a random variable in $\mathbb{R}^d$ with the p.d.f. $f_X < \infty$ such that

$$\|x\| = \|y\| \implies f_X(x) = f_X(y).$$

Let $Z$ be the random variable defined as $Z = X/\|X\|$. Then $Z$ is a uniform sample from the sphere of radius 1 in $\|\cdot\|$.

Hence the problem reduces to finding an appropriate distribution for $X$ that is easy to sample from. (Why?) Devise such distributions for each of $L_1$, $L_2$ and $L_\infty$, and show how to sample from them efficiently given access to random variables distributed uniformly in $[0,1]$.

**Hint:** For example, for $L_2$ the coordinates of $X$ can be independent Gaussians.

**Bonus question:** Does Theorem 0.1 hold for all norms? If not, can you characterize all the norms it holds for?

2. **Convex B-Splines**

Consider a B-spline that is a closed curve such that its control path forms a convex polygon. Prove that the B-spline is the boundary of a convex region.
**Hint:** Use the variation diminishing property of B-splines. The variation diminishing property is that if $P$ is a the control polygon of a curve $B$ then for any straight line $L$ the number of crossing of $P$ by $L$ is at least as large as it is for $C$. You do not need to prove this property for B-splines but you can use this fact.