1. **Geri Plays Chess**

As Introduction to recursive subdivision surfaces you will need to do the following:

1. First watch the movie Geri’s Game:
   
   https://www.youtube.com/watch?v=zWV3XVhaQsc

2. Read the paper from Pixar:
   

   Our you may search for it. The paper is

   Subdivision Surfaces in Character Animation.
   
   
   Pixar Animation Studios.

2. **Optimality of Ruppert’s Algorithm**

   In class we showed that Ruppert’s Delaunay refinement algorithm terminated with all angle at least 20°. The goal of this exercise is to show that up to constants it will generate an optimal size such mesh. We will do this by showing that Ruppert’s Algorithm and any other optimal mesh has a nice characterization.

   For simplicity, in this problem we will only be concerned with meshing sets of points, without line segments. However, a similar result also holds for planar straight line graphs. First, let us state the problem formally.

   **Input.** A set of points \( S \subseteq B \), all of which are contained in some square bounding box \( B \). We assume the corners of \( B \) belong to \( S \).

   **Output.** A set \( T \supseteq S \) such that the Delaunay triangulation of \( T \) has no angles less than 15°.
We want to show that the output of Ruppert’s algorithm is a set of size at most a constant factor greater than optimal possible.

Let us recall the core tool we use to analyze meshings. The local feature size, lfs$_A(x)$ for a point $x \in B$ with respect to a finite set $A \subseteq B$ is simply the distance to the second nearest neighbor in $A$ to $x$.

We have seen in class that the output $R$ of Ruppert’s algorithm guarantees

$$\text{lfs}_R(x) \geq c \cdot \text{lfs}_S(x) \quad \text{for all } x \in B$$

(1)

for some constant $c_0$.

First, we will want to show that

$$|R| \leq c_1 \int_{x \in B} \frac{1}{\text{lfs}_S(x)^2} dx$$

(2)

for some constant $c_1$.

**Subtask 1.** Show that the cells of the Voronoi diagram of any correct output $T$ to our problem have constant aspect ratio (that is, the distance from the center is within a constant factor for all the points on the boundary). It is equivalent to showing that the ratio of out-radius to in-radius is bounded.

*Hint:* The constant may and should be rather big.

**Subtask 2.** Conclude that the Voronoi cell around each $p \in R$ contains a ball of radius $\Omega(lfs_R(p))$. Show that the integral of $\text{lfs}_R(x)^{-2}$ over this ball is constant. Use this and Inequality 1 to prove Inequality 2.

*Hint:* Use that the lfs function is 1-Lipschitz.

Now, let $T$ be any superset of $S$ whose Delaunay triangulation has no angles less than 15°. We want to show that

$$|T| \geq c_2 \int_{x \in B} \frac{1}{\text{lfs}_S(x)^2} dx$$

(3)

for some constant $c_2$.

**Subtask 3.** Show that if $x$ belongs to the Voronoi cell of some $p \in T$, then

$$\text{lfs}_T(x) = \Omega(\text{lfs}_T(p)).$$

**Subtask 4.** Use this to derive Inequality 3.

*Hint 1:* Use the result of Subtask 1.

*Hint 2:* What is the relation between $\text{lfs}_S(x)$ and $\text{lfs}_T(x)$?

**Subtask 5.** Conclude that $|T| = \Omega(\|R\|)$. 

Page 2
3. VC Dimension of Polytopes

**Subtask 1.** Prove that the VC dimension of the set of convex polygons with at most $n$ vertices $\{CC(S) : |S| = n, S \subseteq \mathbb{R}^2\}$ is $O(n)$.

*Hint:* Think of the polygon as an intersection of halfplanes.

**Subtask 2.** Prove that the VC dimension of the set of convex 3D polytopes with at most $n$ vertices $\{CC(S) : |S| = n, S \subseteq \mathbb{R}^3\}$ is $O(n)$.

**Subtask 3.** Derive any upper bound in terms of $n$ and $d$ on the VC dimension of the set of convex $d$-dimensional polytopes with at most $n$ vertices $\{CC(S) : |S| = n, S \subseteq \mathbb{R}^d\}$.

**Open question.** Can you prove a poly($n, d$) bound for the previous problem?