15-451 Algorithms, Spring 2016
Recitation #12 Worksheet

VCG and Pricing Advertisements

We saw the VCG mechanism for incentive-compatible auctions in Lecture. Let's use this for pricing online advertising slots. There are 2 ad slots that ElGogo wants to sell on a page, the first slot has a clickthru rate of 0.5, the second has a clickthru rate of 0.3. *Each bidder can get at most one slot.* There are 4 bidders, with the following valuations:

- A: \$10 per click (so, e.g., this bidder values the first slot at 10*0.5 = 5, and the second slot at 10*0.3 = 3.)
- B: \$8 per click
- C: \$7 per click
- D: \$2 per click

1. What is the social-welfare maximizing allocation?

   Solution: A gets the first slot, B gets the second. The total value to A is 5 and the value to B is 8*0.3 = 2.4. Total social welfare = 7.4.

2. What are the VCG payments?

   Solution: If A did not bid, B and C would get the first and second slots respectively. The social welfare would be 8*0.5 + 7*0.3 = 6.1. So A's payment is how much his presence caused B,C,D's welfare to fall, = (optimal welfare without A) - (optimal welfare of everyone else with A) = 6.1 - (2.4 + 0 + 0) = 3.7.

   If B did not bid, A and C would get the first and second slots respectively. The social welfare would be 10*0.5 + 7*0.3 = 7.1. So B's payment is how much his presence caused the others' welfare to decrease = (optimal welfare without B) - (optimal welfare of everyone else with B) = 7.1 - (5 + 0 + 0) = 2.1.

   C and D do not pay anything.

Combinatorial Auctions

VCG can be used even with complicated preferences. Suppose we have two identical hotel rooms in Las Vegas, a flight ticket f from PIT to LAS, and a concert ticket c in Vegas to auction off. In the following, a generic hotel room is denoted by h, and none of the people want two rooms.

- Buyer A: values $\{h\}$ at \$100, $\{f\}$ at \$200, $\{h, f\}$ at \$450, $\{h, f, c\}$ at \$440. (He hates the band in question so much, he gets *negative value* from getting c along with h, f.) All other sets are valued at \$0.

- Buyer B (doesn't care for the concert): values $\{h\}$ at \$50, $\{f\}$ at \$400, $\{h, f\}$ at \$500, and $\{h, f, c\}$ at \$501. All other sets are valued at \$0.
- Buyer C (lives in Vegas): values $\{c\}$ (and all sets containing c) at \$200.

What is the social-welfare maximizing allocation, and what are the VCG payments?

Solution: The allocation is A gets $\{h\}$, B gets $\{h, f\}$, and C gets $\{c\}$. This gives a total valuation (aka social welfare) of $\$100 + 500 + 200 = \$800$.

A pays 0, B pays \$350, C pays \$1.

Minimax Using MW/WM

We didn't have time to go over the proof that RWM implies the minimax theorem. Here's the proof, to go over in reci.

- Recall that V_R is how much the row player can ensure he will make, no matter what the column player does.
- And V_C is what the column player can restrict the row player to making,, no matter what the row player does.
- So $V_R \leq V_C$. The minimax theorem says these are in fact equal. We'll prove this.

The randomized weighted majority (RWM) can be used to play a zero-sum game: we're given a matrix of payoffs to the adversary. (Say these payoffs are $\{0, 1\}$-valued.) Each day the adversary plays a row, we play a column using RWM. We pay the adversary the number in the resulting entry.

- The guarantee from RWM is that after T days says that the (expected) cost to us is at most the cost incurred by the best column (expert) in hindsight $+\epsilon T + \frac{logn}{\epsilon}$.
- Looking at the best column (expert) in hindsight means the row player went first, and then we choose the best response. This would give row player payoff at most $V_R \cdot T$.
- On the other hand, each day the adversary knows what distribution we're playing from, so it's like she plays second. She can ensure a payoff of at least V_C. I.e., she is guaranteed a payoff of $V_C \cdot T$ over T days.
- So
$$V_C \cdot T \leq \text{ row player's payoff } \leq V_R \cdot T + \epsilon T + \frac{\ln n}{\epsilon},$$
or
$$(V_C - V_R)T \leq \epsilon T + \frac{\ln n}{\epsilon}.$$
And remember, we can choose ϵ in RWM.
- So if $V_R = V_C - \delta$ for some constant δ, we could choose $\epsilon = \delta/2$ and play for $T \geq \frac{\log n}{\epsilon^2}$ days to get a contradiction.