Today, we’ll do some practice problems on amortized analysis.

1. **Queue as Two Stacks**

   We want to implement a queue $Q$ with two stacks $S_1$ and $S_2$, but in a way that is still efficient.

   (a) How would you implement the queue? Specifically, how should the push and pop operations of the queue be implemented with the two stacks?

   **Solution:** We’ll make $S_1$ be the head of the queue and $S_2$ be the “reverse” of the tail of the queue. (Draw picture if necessary).

   Then $Q$ can be implemented as follows. $\text{push}(x, Q)$ will be $\text{push}(S_1, x)$. For $\text{pop}(Q)$, if $S_2 \neq \emptyset$, then $\text{pop}(S_2)$; otherwise, pop every element of $S_1$ and push it into $S_2$ in their respective order (so while $S_1 \neq \emptyset$, $\text{push}(\text{pop}(S_1), S_2)$), and then $\text{pop}(S_2)$.

   (b) Using the accounting/banker’s/token method, find the amortized costs of each operation on $Q$. Assume that a push and a pop each have a cost of 1 (token).

   **Solution:** When we push $x$ onto $Q$, pay 3 tokens: one for the push of $x$ onto $S_1$, and two to put onto the element for future use. This way, every element in $S_1$ will have 2 tokens. When we have to move elements of $S_1$ to $S_2$, we have enough tokens to pay for the move for every element in $S_1$ (one for the pop and one for the push). Finally, for a pop from $Q$, we simply pay one token to pop from $S_2$.

   Overall, at most $4n$ tokens are necessary for $n$ operations, giving an amortized cost of $4 \in O(1)$. More specifically, we saw that a push into $Q$ has an amortized cost of 3 and a pop from $Q$ has an amortized cost of 1.

   (c) Recall the potential method for finding the amortized costs of operations. The potential method is simply another way to look at the accounting method by assigning values (potentials) to particular states of the problem, and then allowing certain costs to be covered by changes in potential. This idea is similar to what one might expect in physics. More precisely:

   We are given a set of possible states $S$ that the given problem can take, and a set of “unit” operations $O$ one can apply to any state; note, these are not necessarily the operations the problem statement only defines (this will be clearer below). Then we define a potential function $\Phi : S \rightarrow \mathbb{R}$ that assigns potentials to the states of the problem. We define the unit cost $UC_f$ of an operation $f \in O$ to be the
raw/immediate cost of actually applying the operation. If we apply a sequence of 
$n$ operations $f_1, \ldots, f_n$, then we define the amortized cost of the sequence of the $n$
operations to be the average of the unit costs of each operation with the change in
potential:

$$AC = \sum_{i=1}^{n} (UC_{f_i} + \Phi(S_i) - \Phi(S_{i-1})) = \sum_{i=1}^{n} UC_{f_i} + \Phi(S_n) - \Phi(S_0)$$

If we are to use the potential method, what would the unit operations in this
problem. What are their respective unit costs? What should we take our potential
function to be?

**Solution:** The unit operations here would be

i. a push into $S_1$ (used in push into $Q$)

ii. a pop from $S_1$ followed by a push into $S_2$ (used in pop from $Q$)

iii. a pop from $S_2$ (used in pop from $Q$)

Take the potential function to be $\Phi(Q) = 2|S_1|$. The unit cost of a pop is
$UC_{pop} = 1$, and the unit cost of a push is $UC_{push} = 1$.

(d) Using the potential method, find the amortized costs of each operation on $Q$.

**Solution:** Consider $push(x, Q)$. This operation consists of exactly one unit
operation: $push(x, S_1)$. $push(x, S_1)$ causes $S_1$ to increase in size by 1, so that
the potential increases by 2, or $\Delta_{push} \Phi = 2$. Hence

$$AC_{Q-push} = UC_{S_1-push} + \Delta_{S_1-push} \Phi = 1 + 2 = 3.$$ 

Now consider $pop(Q)$. This operation consists of exactly $|S_1| + 1$ unit operations:
$|S_1|$ of $push(pop(S_1), S_2)$, and 1 of $pop(S_2)$. For every element in $S_1$ we pop off
and push into $S_2$, the unit cost is 2 and the change in potential is -2:

$$AC_{pop-push} = UC_{pop-push} + \Delta_{pop-push} \Phi = 2 + (-2) = 0.$$ 

The final pop from $S_2$ has a unit cost of 1, and the potential does not change

$$AC_{final-pop} = UC_{final-pop} + \Delta_{final-pop} \Phi = 1 + 0 = 1.$$ 

This implies that the amortized cost of a pop from $Q$ is

$$AC_{Q-pop} = \sum_{i=1}^{|S_1|} (UC_{pop-push} + \Delta_{pop-push} \Phi) + UC_{final-pop} + \Delta_{final-pop} \Phi$$

$$= |S_1| \cdot 0 + 1 = 1.$$
We again see here that a push into $Q$ has an amortized cost of 3 and a pop from $Q$ has an amortized cost of 1.

2. **Redundant Ternary Counter**

In previous classes (15-122, for example), you have seen amortized analysis of the binary counter. One problem with the binary counter is that you cannot both increment and decrement with the same amortized $O(1)$ guarantee for each increment or decrement. Here, we introduce the redundant ternary counter, where each digit may be $-1, 0$ or 1. The underlying base is still 2.

For example, we can represent 3 as $11_2$, or $10(-1)_2$.

The process of incrementing a ternary number is analogous to that operation on binary numbers. You add 1 to the low order trit. If the result is 2 then it is changed to 0, and a carry of 1 is propagated to the next trit. This process is repeated until no carry results. Decrementing a number is similar. You subtract 1 from the low order trit. If it becomes $-2$ then it is replaced by 0, and a carry of $(-1)$ is propagated.

(a) Using the accounting/banker’s/token method, find the amortized costs of an increment and a decrement. Assume the cost of changing the value of a trit is 1 (token).

**Solution:** Note that when the number is incremented or decremented, a 1 can only change to a 0 or stay the same; similarly, a $-1$ can only change to a 0 or stay the same. This seems to suggest that if we pay 2 tokens each time a trit is changed from a 0, the extra token will cover the cost of converting the trit back to a 0.

Finally, for any one increment/decrement, note that the number of trits that change a 0 to a non-zero value is at most 1. (Why? Do an example).

Hence, the amortized cost of both an increment and a decrement is at most 2.

(b) Using the potential method, find the amortized costs of each increment and decrement.

**Solution:** The unit operations here would be simply changing a trit’s value. Take the potential function to be $\Phi = \text{(number of non-zero trits)}$. The unit cost of a flip of a trit is $UC_{\text{flip}} = 1$.

Consider the increment (decrement is analogous). As we say above, at most one 0 is changed. Suppose the operation consists of exactly $k$ flips of non-zero trits and $m$ flips of a 0 trit, where $m \leq 1$, so the total unit costs is $k + m$. Note then
that the total change in potential is $\Delta \Phi = m - k$. Hence

$$AC_{inc} = (k + m) + (m - k) = 2m \leq 2.$$ 

We again see here that an increment and decrement both at amortized cost at most 2.