15-451 — Algorithms — Spring 2006

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Homework #5

Due: April 6, 2006, start of class.

Some Reminders:

- You may discuss these problems with others, in small groups. However we strongly recommend that you think for a while about them yourself before starting such discussions.
- The work that you turn in must be your own, written by you in your own words. We are allowing handwritten solutions, although typeset ones are preferred. If you handwrite, WRITE CLEARLY, or we will revert to the old system of requiring you to typeset solutions.
- The cover page of your submission must clearly display the assignment number, your name, your recitation section and your Andrew ID.

Problems:

1 A Bipartite Game

Consider the following 2-player perfect information game. The two players are called Left and Right. The game is played on an undirected bipartite graph G, in which the two parts are called L and R. Left goes first by picking any vertex of L. Then Right picks a neighbor of this vertex (which is in R, of course). Play continues in this way, with each player picking a vertex of the appropriate set neighboring the vertex just picked, with the additional proviso that a vertex may be used only once. The player who finds herself without a legal move loses the game.

(a) Assume that the graph G has a matching in which every vertex of L is in the matching. Prove that in this case Right has a winning strategy.

(b) Assume that the graph G has a matching in which every vertex of R is in the matching, but some vertex of L is not in the matching. Prove that in this case Left has a winning strategy.

(c) Extend parts (a) and (b) to give a complete characterization of the class of graphs on which Left has a winning strategy. Explain why your characterization is correct.

(d) Now give a polynomial time algorithm which takes an arbitrary bipartite graph G and determines which player has a winning strategy.

2 Cohesion

Consider problem 46 on page 444 of the book. The solution to the problem is to build a graph H from G, and to find a min cut in H. This graph has the property that if the minimum cut (in a certain class of cuts) is sufficiently small, then the cohesion is at least α.

Here's how we build H. The graph has four columns of vertices. The left column (column 1) is the source s. The next column has a vertex for each vertex of G. The third column has a vertex for each edge of G, and the fourth column is t. The edges go from a column to the next column to the right. The edges from s to column 2 all have capacity α. The edges from column 2 to column 3 all have infinite capacity, and there is such an edge if the corresponding vertex and edge of G are incident. The edges from column 3 to t all have capacity 1.

Let n be the number of vertices of G, and let m be the number of edges of G.

(a) Now prove the theorem:

> Theorem: The graph G has cohesion at least α if and only if the minimum cut of H (among all the cuts which cut at least one of the edges from column 1 to column 2) is at most m.

(b) Give a polynomial-time algorithm that can compute the appropriate minimum cut of H that's needed in the theorem.