Introduction

In this short chapter, we shall explain what is meant by linear programming and sketch a history of this subject.

A DIET PROBLEM

Polly wonders how much money she must spend on food in order to get all the energy (2,000 kcal), protein (55 g), and calcium (800 mg) that she needs every day. (For iron and vitamins, she will depend on pills. Nutritionists would disapprove, but the introductory example ought to be simple.) She chooses six foods that seem to be cheap sources of the nutrients; her data are collected in Table 1.1.

<table>
<thead>
<tr>
<th>Food</th>
<th>Serving size</th>
<th>Energy (kcal)</th>
<th>Protein (g)</th>
<th>Calcium (mg)</th>
<th>Price per serving (cents)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oatmeal</td>
<td>28 g</td>
<td>110</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Chicken</td>
<td>100 g</td>
<td>205</td>
<td>32</td>
<td>12</td>
<td>24</td>
</tr>
<tr>
<td>Eggs</td>
<td>2 large</td>
<td>160</td>
<td>13</td>
<td>54</td>
<td>13</td>
</tr>
<tr>
<td>Whole milk</td>
<td>237 cc</td>
<td>160</td>
<td>8</td>
<td>285</td>
<td>9</td>
</tr>
<tr>
<td>Cherry pie</td>
<td>170 g</td>
<td>420</td>
<td>4</td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>Pork with beans</td>
<td>260 g</td>
<td>260</td>
<td>14</td>
<td>80</td>
<td>19</td>
</tr>
</tbody>
</table>
Then she begins to think about her menu. For example, 10 servings of pork with beans would take care of all her needs for only (?) $1.90 per day. On the other hand, 10 servings of pork with beans is a lot of pork with beans—she would not be able to stomach more than 2 servings a day. She decides to impose servings-per-day limits on all six foods:

- Oatmeal: at most 4 servings per day
- Chicken: at most 3 servings per day
- Eggs: at most 2 servings per day
- Milk: at most 8 servings per day
- Cherry pie: at most 2 servings per day
- Pork with beans: at most 2 servings per day

Now, another look at the data shows Polly that 8 servings of milk and 2 servings of cherry pie every day will satisfy the requirements nicely and at a cost of only $1.12. In fact, she could cut down a little on the pie or the milk or perhaps try a different combination. But so many combinations seem promising that one could go on and on, looking for the best one. Trial and error is not particularly helpful here. To be systematic, we may speculate about some as yet unspecified menu consisting of $x_1$ servings of oatmeal, $x_2$ servings of chicken, $x_3$ servings of eggs, and so on. In order to stay below the upper limits, that menu must satisfy

\begin{align}
0 & \leq x_1 \leq 4 \\
0 & \leq x_2 \leq 3 \\
0 & \leq x_3 \leq 2 \\
0 & \leq x_4 \leq 8 \\
0 & \leq x_5 \leq 2 \\
0 & \leq x_6 \leq 2.
\end{align}  \tag{1.1}

And, of course, there are the requirements for energy, protein, and calcium; they lead to the inequalities

\begin{align}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 & \geq 2,000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 & \geq 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 & \geq 800.
\end{align}  \tag{1.2}

If some numbers $x_1, x_2, \ldots, x_6$ satisfy inequalities (1.1) and (1.2), then they describe a satisfactory menu; such a menu will cost, in cents per day,

\begin{align}
3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6.
\end{align}  \tag{1.3}

In designing the most economical menu, Polly wants to find numbers $x_1, x_2, \ldots, x_6$ that satisfy (1.1) and (1.2), and make (1.3) as small as possible. As a mathematician
would put it, she wants to

\[
\begin{align*}
\text{minimize} & \quad 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{subject to} & \quad 0 \leq x_1 \leq 4 \\
& \quad 0 \leq x_2 \leq 3 \\
& \quad 0 \leq x_3 \leq 2 \\
& \quad 0 \leq x_4 \leq 8 \\
& \quad 0 \leq x_5 \leq 2 \\
& \quad 0 \leq x_6 \leq 2 \\
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 & \geq 2000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 & \geq 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 & \geq 800.
\end{align*}
\]

(1.4)

Her problem is known as a \textit{diet problem}.

**LINEAR PROGRAMMING**

Problems of this kind are called “linear programming problems,” or “LP problems” for short; linear programming is the branch of applied mathematics concerned with these problems. Here are other examples:

\[
\begin{align*}
\text{maximize} & \quad 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} & \quad 2x_1 + 3x_2 + x_3 \leq 5 \\
& \quad 4x_1 + x_2 + 2x_3 \leq 11 \\
& \quad 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

(with “\(x_1, x_2, x_3 \geq 0\)” used as shorthand for “\(x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\)” or

\[
\begin{align*}
\text{minimize} & \quad 3x_1 - x_2 \\
\text{subject to} & \quad -x_1 + 6x_2 - x_3 + x_4 \geq -3 \\
& \quad 7x_2 + 2x_4 = 5 \\
& \quad x_1 + x_2 + x_3 = 1 \\
& \quad x_3 + x_4 \leq 2 \\
& \quad x_2, x_3 \geq 0.
\end{align*}
\]

(1.5)

(1.6)

In general, if \(c_1, c_2, \ldots, c_n\) are real numbers, then the function \(f\) of real variables \(x_1, x_2, \ldots, x_n\) defined by

\[
f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{j=1}^{n} c_jx_j
\]