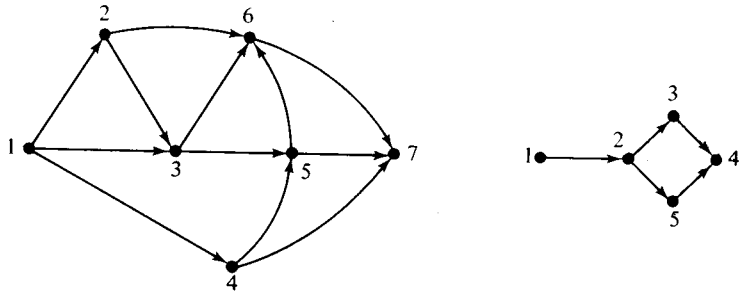


(a) Topologically sorted. (b) Not topologically sorted.

Figure 8.6 Acyclic digraphs.

incoming edges. Since the remaining digraph is also acyclic, we can repeat the process and assign the next highest number, namely $|V| - 1$, to a vertex with no outgoing edges, and so on. To keep the algorithm $O(|V| + |E|)$, we must avoid searching the modified digraph for a vertex with no outgoing edges.

We do so by performing a single depth-first search on the given acyclic digraph G. In addition to the usual array *num*, we will need another array, *label*, of size $|V|$ for recording the topologically sorted vertex labels. That is, if there is an edge (u, v) in G, then *label* (u) < *label* (v). The complete search and labeling procedure *TOPSORT* is given in Algorithm 8.4.

The time required by Algorithm 8.4 is $O(|V| + |E|)$, since every edge is traversed once and the procedure *TOPSORT* is called once for each vertex.

8.2.3 Biconnectivity

Sometimes it is not enough to know that a graph is connected; we may need to know how "well connected" a connected graph is. A connected graph, for example, may contain a vertex whose removal, along with its incident edges, disconnects the remaining vertices. Such a vertex is called an *articulation point* or a *cut-vertex*. A graph that contains an articulation point is called *separable*. Vertices *b*, *f*, and *i* in Figure 8.7(a), for example, are articulation points, and they are the only ones in the graph. A graph with no articulation points is called *biconnected* or *nonseparable*. A maximal biconnected subgraph of a graph is called a *biconnected component* or a *block*. Identification of the articulation points and biconnected components of a given graph is important in the study of the vulnerability of communication and transportation networks. It is also important in determining other properties, like planarity, of a graph G, since it is often advantageous to separate G into its biconnected components and examine each one individually (see Section 8.6).

A vertex v in an undirected connected graph is an articulation point if and only if there exist two other vertices x and y such that every path between x and y passes through v; in this case and only in this case does the deletion of v from G destroy all paths between x and y (i.e., disconnect G). This observation allows us

```
for x ∈ V do { num (x) ← 0
             { label (x) ← 0
j ← |V| + 1
i ← 0
for x ∈ V do if num (x) = 0 then { [[x has no labeled ancestor]]
                                 { TOPSORT (x)
procedure TOPSORT (v)
    i ← i + 1
    num (v) ← i
    for w ∈ Adj (v) do { [[process all descendants of w]]
                       { if num (w) = 0 then TOPSORT (w)
                       {      else if label (w) = 0 then [[G has a cycle]]
    [[v is labeled with a value less than
     the value assigned to any descendant]]
    j ← j − 1
    label (v) ← j
    return
```

Algorithm 8.4 Topological sort of an acyclic digraph $G = (V, E)$.

to use depth-first search to find the articulation points and biconnected components of a graph in $O(|V| + |E|)$ operations.

The central idea can be understood by studying the example in Figure 8.8, which shows schematically a connected graph consisting of biconnected components G_i, $1 \leqq i \leqq 9$, and articulation points v_j, $1 \leqq j \leqq 5$. If we start the depth-first search at, say, the vertex s in G_9, we might, perhaps, leave G_9 to go into G_4 by

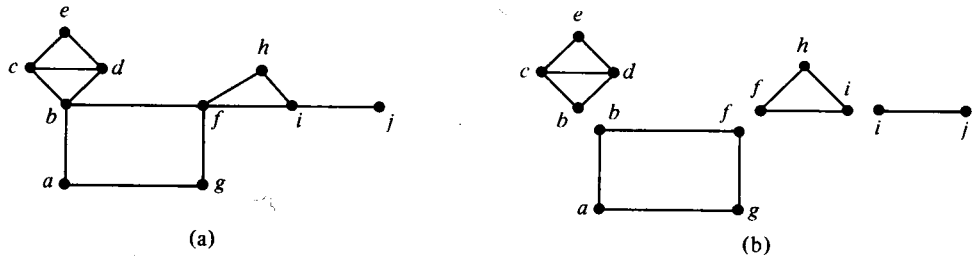


Figure 8.7 A separable graph (a) and its biconnected components (b). The articulation points are *b*, *f*, and *i*.

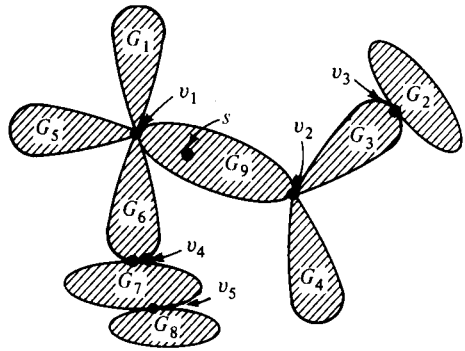


Figure 8.8 A schematic drawing of a graph with nine biconnected components and five articulation points.

passing through v_2. But by the depth-first nature of the search, all the edges in G_4 must be traversed before we back up to v_2; thus G_4 consists of exactly the edges traversed between visits to v_2. Matters are actually a little more complicated for the other biconnected components, since, for example, if we leave G_4 and go into G_3 and from there into G_2 through v_3, we would find ourselves in G_2, having traversed edges from G_9, G_3, and G_2. Fortunately, however, if we store the edges in a stack, by the time we pass through v_3 back into G_3 all the edges of G_2 will be on top of the stack. When they are removed, the edges on top of the stack will be from G_3, and we will once again be traversing G_3. Thus if we can recognize the articulation points, we can determine the biconnected components by applying depth-first search and storing the edges on a stack as they are traversed; the edges on top of the stack as we back up through an articulation point form a biconnected component.

In order to recognize an articulation point we need to compute, during the depth-first search, a new function *lowpt* (v) for every vertex v in the graph. We define *lowpt* (v) as the smallest value of *num* (x), where x is a vertex of the graph that can be reached from v by following a sequence of zero or more tree edges followed by at most one back edge. The function *lowpt* (v) is useful because of the following theorem.

Theorem 8.1

Let $G = (V, E)$ be a connected graph with a DFS-tree T and with back edges B. Then $a \in V$ is an articulation point if and only if there exist vertices v, $w \in V$ such that $(a, v) \in T$, w is not a descendant of v in T and *lowpt* $(v) \geqq$ *num* (a).

Proof: Suppose that such vertices v and w exist. Since $(a, v) \in T$ and *lowpt* $(v) \geqq$ *num* (a), any path starting at v that does not go through a must remain in the subtree with root v. Since w is not a descendant of v in T, such a path cannot contain w. Thus the only paths from v to w contain a and so a is an articulation point.

Conversely, suppose that a is an articulation point. If a is the root of T, then at least two edges of T start at a; otherwise there would be a path in G between every pair of vertices in $V - \{a\}$ that did not contain a. Let (a, v) and (a, w) be two of these edges; clearly, v and w satisfy the theorem. If a is not the root of T, then it has an ancestor w. One of the biconnected components containing a has all its nodes as descendants of a in T; in fact, they are all (except a) descendants of a vertex v; where (a, v) is an edge in T (why?). Clearly, v and w satisfy the theorem.

This theorem tells how to recognize articulation points if we have the values of *num* and *lowpt*: if we find a vertex v such that $(a, v) \in T$ and $lowpt\,(v) \geqq num\,(a)$, then a is either an articulation point or the root of T. This result follows from Theorem 8.1 by observing that a suitable w can be chosen among the ancestors of a if a is not the root. Furthermore, computing the *lowpt* values during depth-first search is simple because

$$lowpt\,(v) = \min(\{num\,(v)\} \cup \{lowpt\,(x) | (v, x) \in T\} \cup \{num\,(x) | (v, x) \in B\}).$$

Thus Algorithm 8.5 determines the biconnected components of a graph $G = (V, E)$. Since the algorithm is a depth-first search with a constant amount of extra work done as each edge is traversed, the time required is clearly $O(|V| + |E|)$. A proof that the algorithm works correctly is left as Exercise 20.

8.2.4 Strong Connectivity

In digraphs, the properties involving paths, cycles, and connectivity become more complicated than in undirected graphs because of the orientations of the edges. The (weakly) connected components of a digraph are easily obtained by ignoring the edge directions and using Algorithm 8.3 to find the connected components of the resulting undirected graph. However, we are usually more interested in finding the *strongly connected components*—that is, the maximal strongly connected subgraphs. A digraph and its strongly connected components are illustrated in Figure 8.9.

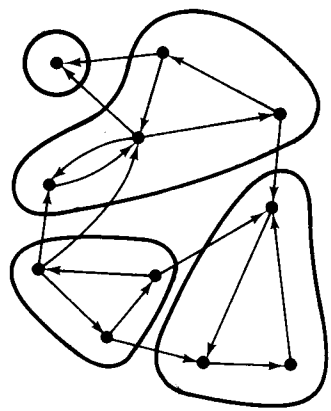


Figure 8.9 A digraph with its strongly connected components outlined in boldface.

$i \leftarrow 0$

$S \leftarrow$ empty stack

for $x \in V$ **do** $num\ (x) \leftarrow 0$

for $x \in V$ **do if** $num\ (x) = 0$ **then** $BICON\ (x, 0)$

procedure $BICON\ (v, u)$

$i \leftarrow i + 1$

$num\ (v) \leftarrow i$

$lowpt\ (v) \leftarrow i$

for $w \in$ Adj (v) **do if** $num\ (w) = 0$ **then**
- [[(v, w) is a tree edge]]
- $S \Leftarrow (v, w)$
- $BICON\ (w, v)$
- $lowpt\ (v) \leftarrow \min\ (lowpt\ (v), lowpt\ (w))$
- **if** $lowpt\ (w) \geqq num\ (v)$ **then**
  - At this point v is either the root of the tree or it is an articulation point. Form a new biconnected component consisting of all the edges on the stack above and including (v, w). Remove these edges from the stack.

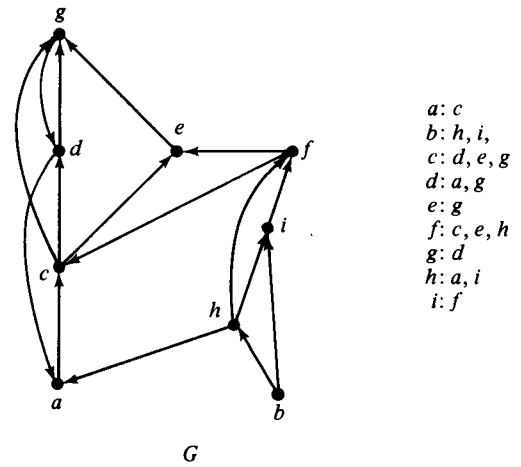
else if $num\ (w) < num\ (v)$ **and** $w \neq u$ **then**
- [[(v, w) is a back edge]]
- $S \Leftarrow (v, w)$
- $lowpt\ (v) \leftarrow \min\ (lowpt\ (v), num\ (w))$

return

Algorithm 8.5 Determining the biconnected components of $G = (V, E)$.

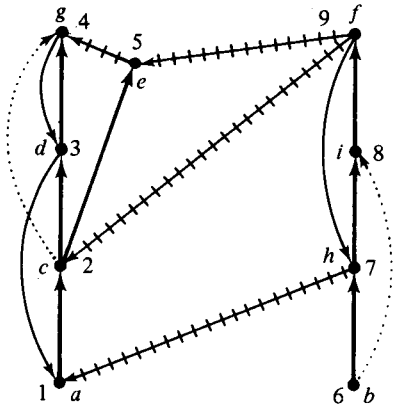
Depth-first search can be applied to digraphs to determine the strongly connected components, but the resulting structure will be more complicated than a set of tree edges and a set of back edges. This complication occurs because the edges in a digraph are already oriented, and we are not free to traverse them in the direction of our choosing, as we did in the depth-first search on an undirected graph.

Consider what happens when we are constrained to traverse the edges of a digraph G along their orientations during a depth-first search on G. As before, we assign a serial number *num* (x) to each vertex x the first time that we visit it. These numbers are permanently assigned to the vertices and indicate the order



G

(a) A digraph G and its adjacency structure.



(b) Digraph G after a depth-first search on it.

Figure 8.10 A digraph (a) traversed by depth-first search. The result (b) consists of 2 trees (boldface), 3 back edges (lightface), 4 cross edges (crossed lines), and 2 forward edges (dotted lines).