Recap of this week’s lectures:

- Zero-Sum Game
- FFT

Minimax from Duality (by Example). Let the row-player’s payoffs be given by this (non-negative) matrix

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>R</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

1. If the probabilities on the two rows are $p_1$ and $p_2$, write down an LP for the row player’s optimal strategy, assuming the column player knows your strategy:

**Solution:** If the row player puts $p_1 \geq 0$ and $p_2 \geq 0$ on $L, R$ respectively, then she wants to solve $\max_{p_1+p_2=1, p_1, p_2 \geq 0} (p_1 + 3p_2, 5p_1 + 2p_2)$. I.e., the LP is

$$\begin{align*}
\max v \\
\text{subject to} \quad & p_1 + 3p_2 \geq v \\
& 5p_1 + 2p_2 \geq v \\
& p_1 + p_2 \leq 1 \\
& p_1, p_2 \geq 0.
\end{align*}$$

We set $p_1 + p_2 \leq 1$, but to maximize $v$, the LP will automatically set the sum equal to 1.

2. Now take the dual of this LP. Show this dual is an LP computing the column player’s optimal strategy. (And hence strong duality implies the minimax theorem.)

**Solution:** First convert into inequalities $blah \leq blah$ useful to show an upper bound. (Also, since all payoffs are non-negative, we can add in non-negativity for $v$.

$$\begin{align*}
\max v \\
\text{subject to} \quad & v - p_1 - 3p_2 \leq 0 \\
& v - 5p_1 - 2p_2 \leq 0 \\
& p_1 + p_2 \leq 1 \\
& v, p_1, p_2 \geq 0.
\end{align*}$$

If the dual variables are $q_1, q_2$ and $w$, we get

$$\begin{align*}
\min w \\
\text{subject to} \quad & q_1 + q_2 \geq 1 \\
& -q_1 - 5q_2 + w \geq 0 \\
& -3q_1 - 2q_2 + w \geq 0 \\
& w, q_1, q_2 \geq 0.
\end{align*}$$
Now move some variables around:

\[
\begin{align*}
\min \ w \\
\text{subject to} \quad q_1 + q_2 &\geq 1 \\
\quad w &\geq q_1 + 5q_2 \\
\quad w &\geq 3q_1 + 2q_2 \\
\quad w, q_1, q_2 &\geq 0
\end{align*}
\]

And again observe that to minimize the value, any optimal solution will reduce \( q_1, q_2 \) to make their sum equal to 1. So the LP is solving:

\[
\min_{q_1 + q_2 = 1, q_1, q_2 \geq 0} \max(q_1 + 5q_2, 3q_1 + 2q_2).
\]

That’s the column player’s strategy!!! And by strong duality, we get the minimax theorem for this particular game. Exactly the same idea holds in general, details are in the lecture notes.
String Matching using FFT

1. Suppose we are given a text \( t = t_0t_1 \ldots t_{n-1} \) and a pattern \( p = p_0p_1 \ldots p_{m-1} \). Find all occurrences of \( p \) in \( t \) in \( O(n \log n) \) time using FFT.

**Solution:** Let \( X[i] = \sum_{j=0}^{m-1} (t_{i+j} - p_j)^2 \) for all \( 0 \leq i < n - m \). There is a match at index \( i \) if and only if \( X[i] = 0 \). Now \( X[i] = \sum_{j=0}^{m-1} (t_{i+j} - p_j)^2 = \sum_{j=0}^{m-1} (p_j^2 - 2p_j t_{i+j} + t_{i+j}^2) \).

So define \( Y[i] = \sum_{j=0}^{m-1} p_j t_{i+j} \) for all \( 0 \leq i < n - m \). It remains to calculate \( Y \).

Let \( t' \) be the reverse of \( t \). Then

\[
Y[i] = \sum_{j=0}^{m-1} p_j t_{i+j} = \sum_{j=0}^{m-1} p_j t'_{n-1-i+j}
\]

Let \( k = n - 1 - i - j \). Then

\[
Y[i] = \sum_{j+k=n-1-i} p_j t'_k
\]

This can now be calculated using FFT. Let \( f \) be the polynomial with coefficients \( p_1, p_2, \ldots, p_{m-1} \) and \( g \) be the polynomial with coefficients \( t'_0, t'_1, \ldots, t'_{n-1} \). Then \( Y[i] \) is the coefficient of \( x^{n-1-i} \) in \( f \cdot g \).

2. Now suppose \( p \) contains wildcard characters, which can match any character \( t \). Finds all matches of \( p \) in \( t \) in \( O(n \log n) \) time.

**Solution:** Replace each wildcard with 0. Let

\[
X[i] = \sum_{j=0}^{m-1} t_{i+j} p_j (t_{i+j} - p_j)^2
\]

Then there is a match at index \( i \) if and only if \( X[i] = 0 \). The rest is the same as in part (a).

Specifically, suppose we want to calculate \( Y[i] = \sum_{j=0}^{m-1} p_j^{a} t_{i+j}^{b} \) for some constants \( a \) and \( b \). Let \( f \) be the polynomial with coefficients \( p_1^a, p_2^a, \ldots, p_{m-1}^a \) and \( g \) be the polynomial with coefficients \( t_0^b, t_1^b, \ldots, t_{n-1}^b \). Then \( Y[i] \) is the coefficient of \( x^{n-1-i} \) in \( f \cdot g \).
3. Suppose $p$ contains no wildcards. Compute, in $O(n \log n)$ time, for each index $0 \leq i < n - m$, the number of characters that agree between $p$ and $t[i, i + m - 1]$.

**Solution:**

Let’s solve an easier version of this problem, where for each $i$, we want to calculate the number of agreements between $p$ and $t[i, i + m - 1]$ where both letters are 'a'.

Replace each letter in $p$ and $t$ with 1 if it is an 'a', and 0 otherwise.

Again define $Y[i] = \sum_{j=0}^{m-1} p_j t_{i+j}$ for all $0 \leq i < n - m$. Clearly $Y[i]$ is the value we want for index $i$. $Y$ can be computed with FFT as in part a.

Now repeat for each of the other 25 letters of the alphabet. Sum together all $Y$ arrays to obtain the final answer.
Another 2-Row Game

Suppose the pay-off matrix for the row player is given as such:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

1. What is the optimal strategy for the row player.

**Solution:** Let $p$ be the probability of playing 1. Consider the lines $4 - p$, $1 + p$, $3 - 2p$, and $0 + 3p$.

Then $p = \frac{2}{3}$ and the max value for the row player is $\frac{5}{3}$.

2. What is the optimal strategy for the column player

**Solution:** The column player will only play B and C. We look at the intersection of $2 - p$ and $1 + 2p$ where $p$ is the probability of playing C. Then $p = \frac{1}{3}$. The value for the column player is still $\frac{5}{3}$.