NP-completeness

Slides by Carl Kingsford

The class **P**

P is the set of decision problems whose memberships are decidable by a Turing Machine that makes a polynomial number of steps.

By the Church-Turing thesis, this is the "same" as:

P is the set of decision problems that can be decided by a computer in a polynomial time.

You can just think of your normal computer as a Turing Machine — and we won't worry too much about that formalism.

Efficient Certification

Def. An algorithm V is an efficient certifier for decision problem X if:

1. V is a polynomial time algorithm that takes two input strings I (instance of X) and C (a certificate).
2. V outputs either `yes` or `no`.
3. There is a polynomial $p(n)$ such that for every string I:

   $I \in X$ if and only if there exists string C of length $\leq p(|I|)$ such that $V(I, C) =$ `yes`.

V is an algorithm that can decide whether an instance I is a `yes` instance if it is given some "help" in the form of a polynomially long certificate.

The class NP

NP is the set of languages for which there exists an efficient certifier.

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NP is the set of languages for which there exists an efficient certifier.

P is the set of languages for which there exists an efficient certifier that ignores the certificate.

A problem is in **P** if we can decided it in polynomial time. It is in **NP** if we can decide it in polynomial time, if we are given the right certificate.

$\mathbf{P} \subseteq \mathbf{NP}$

Theorem. $\mathbf{P} \subseteq \mathbf{NP}$

Proof. Suppose $X \in \mathbf{P}$. Then there is a polynomial-time algorithm A for X.

To show that $X \in \mathbf{NP}$, we need to design an efficient certifier $B(I, C)$.

Just take $B(I, C) = A(I)$. $\square$

Every problem with a polynomial time algorithm is in **NP**.

$\mathbf{P} \neq \mathbf{NP}$?

The big question:

$\mathbf{P} = \mathbf{NP}$?

We know $\mathbf{P} \subseteq \mathbf{NP}$. So the question is:

Is there some problem in **NP** that is not in **P**?

Seems like the power of the certificate would help a lot.
But no one knows. . . .

Reductions

Def. Problem X is polynomial-time reducible to problem Y if

- there is a polynomial-time algorithm A
- that converts instances of X into instances of Y such that
- for all I:

$$A(I) = \texttt{yes} \iff I = \texttt{yes}$$

We denote this by $X \leq_P Y$.

Reductions for Hardness

Theorem. *If $Y \leq_P X$ and Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.*

Why? If we *could* solve X in polynomial time, then we'd be able to solve Y in polynomial time using the reduction, contradicting the assumption.

So: If we could find one hard problem Y, we could prove that another problem X is hard by reducing Y to X.

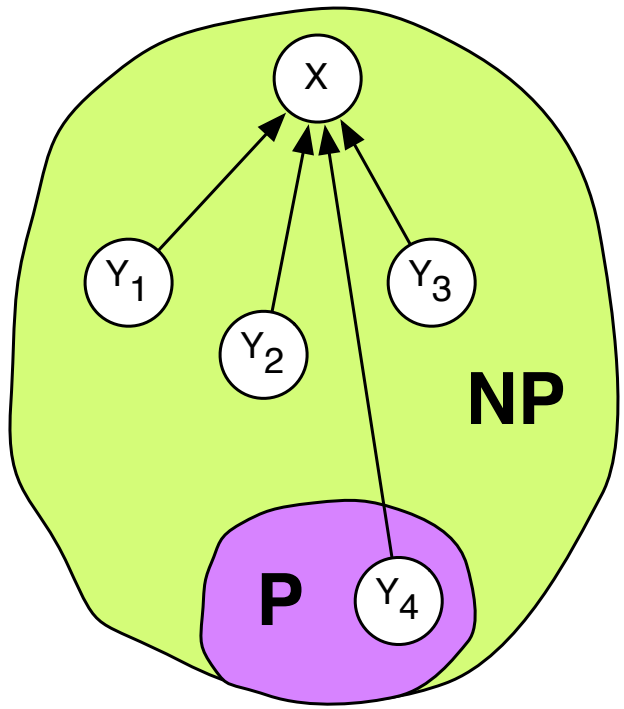
NP-completeness

Def. We say X is NP-complete if:

- $X \in \mathbf{NP}$
- for all $Y \in \mathbf{NP}$, $Y \leq_P X$.

If these hold, then X can be used to solve every problem in **NP**.

Therefore, X is definitely at least as hard as every problem in **NP**.



NP-completeness and P=NP

Theorem. *If X is NP-complete, then X is solvable in polynomial time if and only if* $\mathbf{P} = \mathbf{NP}$.

Proof. If $\mathbf{P} = \mathbf{NP}$, then X can be solved in polytime.

Suppose X is solvable in polytime, and let Y be any problem in **NP**. We can solve Y in polynomial time: reduce it to X.

Therefore, every problem in **NP** has a polytime algorithm and $\mathbf{P} = \mathbf{NP}$.

Reductions and NP-completeness

Theorem. *If Y is NP-complete, and*

1. *X is in NP*
2. *$Y \leq_P X$*

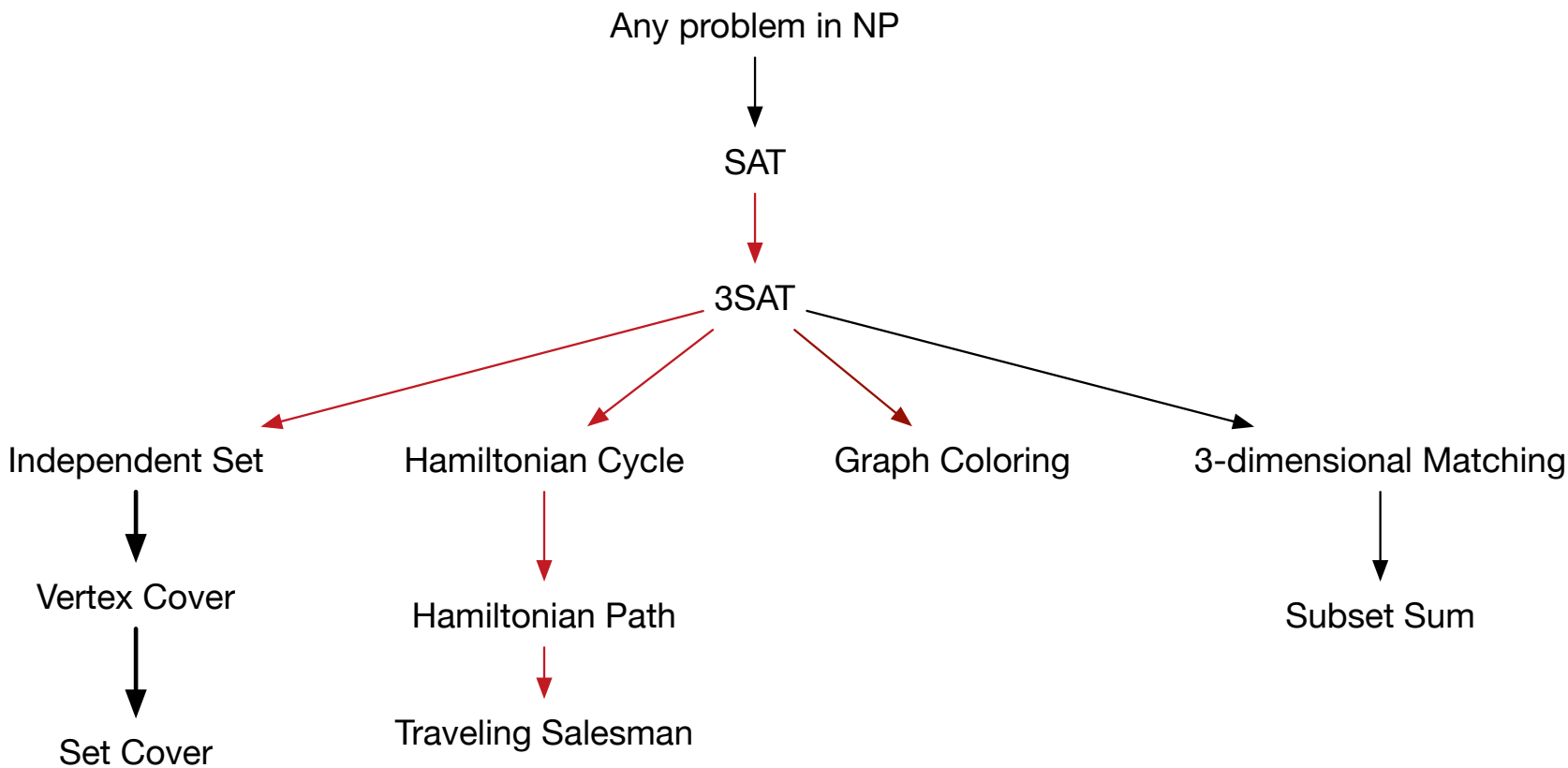
then X is NP-complete.

In other words, we can prove a new problem is NP-complete by reducing some other NP-complete problem to it.

Proof. Let Z be any problem in **NP**. Since Y is NP-complete, $Z \leq_P Y$. By assumption, $Y \leq_P X$. Therefore: $Z \leq_P Y \leq_P X$. $\square$

Chain of Reductions

Cook-Levin Theorem: The problem SAT is NP-complete.



Boolean Formulas

Boolean Formulas:

Variables: x_1, x_2, x_3 (can be either **true** or **false**)

Terms: $t_1, t_2, \ldots, t_\ell$: t_j is either x_i or $\bar{x}_i$
(meaning either x_i or **not** x_i).

Clauses: $t_1 \vee t_2 \vee \cdots \vee t_\ell$ ($\vee$ stands for "OR")
A clause is **true** if any term in it is **true**.

Example 1: $(x_1 \vee \bar{x_2}), (\bar{x_1} \vee \bar{x_3}), (x_2 \vee \bar{v_3})$

Example 2: $(x_1 \vee x_2 \vee \bar{x_3}), (\bar{x_2} \vee x_1)$

Boolean Formulas

Def. A truth assignment is a choice of **true** or **false** for each variable, ie, a function $v : X \to \{\textbf{true}, \textbf{false}\}$.

Def. A CNF formula is a conjunction of clauses:

$$C_1 \wedge C_2, \wedge \cdots \wedge C_k$$

Example: $(x_1 \vee \bar{x_2}) \wedge (\bar{x_1} \vee \bar{x_3}) \wedge (x_2 \vee \bar{v_3})$

Def. A truth assignment is a satisfying assignment for such a formula if it makes every clause **true**.

SAT and 3-SAT

Problem (Satisfiability (SAT)). *Given a set of clauses $C_1, \ldots, C_k$ over variables $X = \{x_1, \ldots, x_n\}$ is there a satisfying assignment?*

Problem (Satisfiability (3-SAT)). *Given a set of clauses $C_1, \ldots, C_k$, each of length 3, over variables $X = \{x_1, \ldots, x_n\}$ is there a satisfying assignment?*

Cook-Levin Theorem

Theorem (Cook-Levin). *SAT is NP-complete.*

Proven in early 1970s by Cook. Slightly different proof by Levin independently.

Idea of the original proof: encode the workings of a Nondeterministic Turing machine for an instance I of problem $X \in \mathbf{NP}$ as a SAT formula so that the formula is satisfiable if and only if the nondeterministic Turing machine would accept instance I.

Another intuition why this is true: A computer is just a circuit, and SAT encodes a kind circuit.

Reducing 3-SAT to Independent Set

Thm. 3-SAT $\leq_P$ Independent Set

Proof. Suppose we have an algorithm to solve Independent Set, how can we use it to solve 3-SAT?

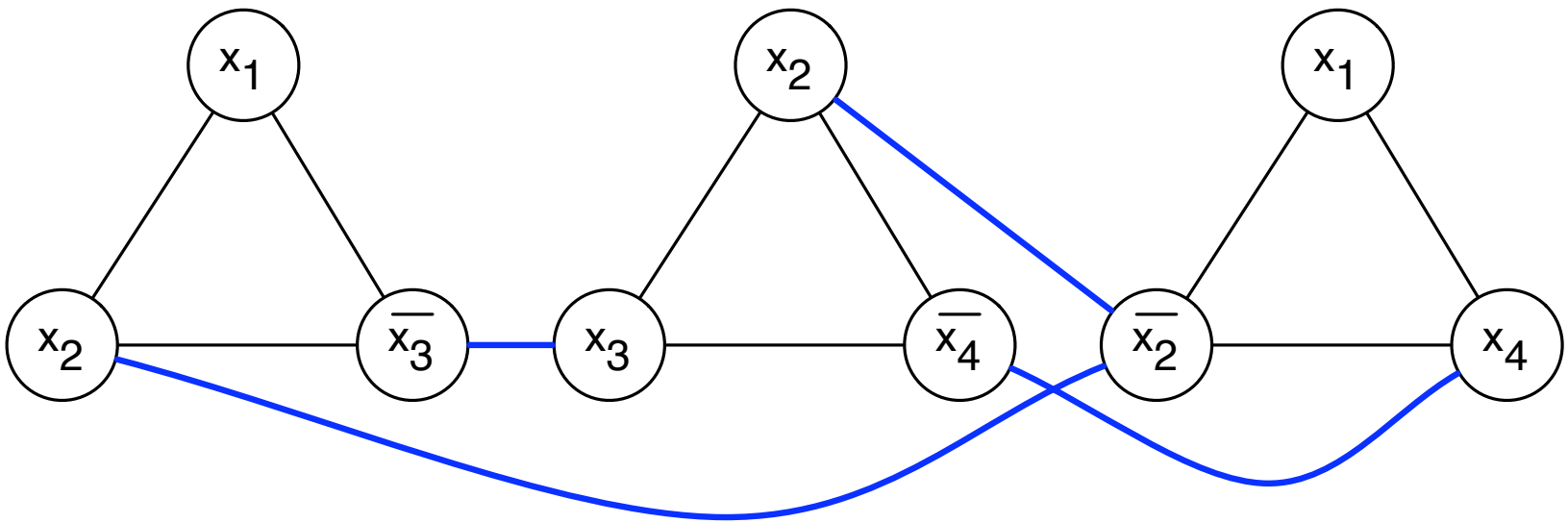
To solve 3-SAT:

- you have to choose a term from each clause to set to **true**,
- but you can't set both x_i and $\bar{x}_i$ to **true**.

How do we do the reduction?

3-SAT $\leq_P$ Independent Set

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_2} \vee x_4)$$



Proof

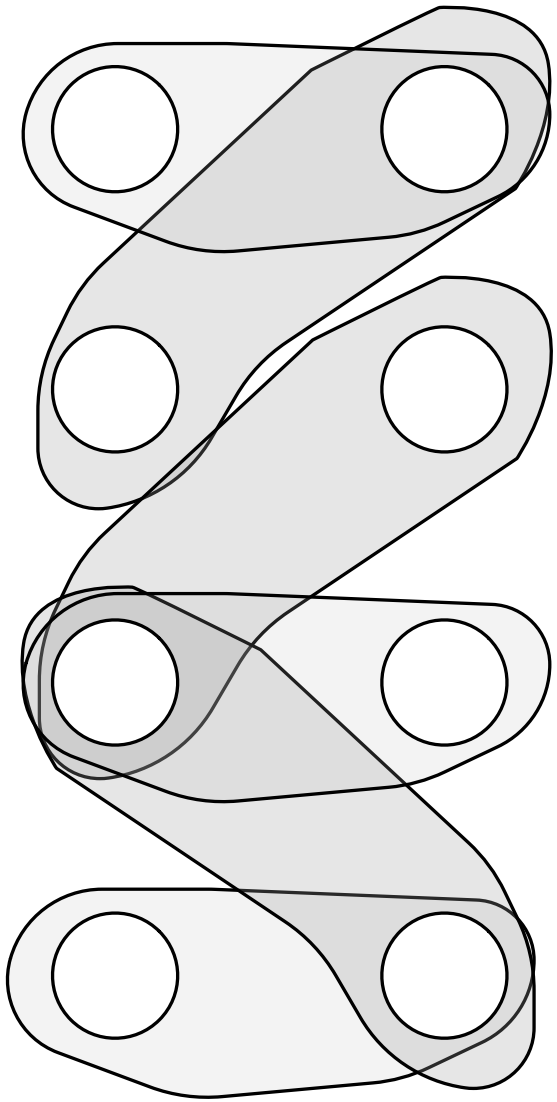
Theorem. *This graph has an independent set of size k iff the formula is satisfiable.*

Proof. $\Longrightarrow$ If the formula is satisfiable, there is at least one true literal in each clause. Let S be a set of one such true literal from each clause. $|S| = k$ and no two nodes in S are connected by an edge.

$\Longrightarrow$ If the graph has an independent set S of size k, we know that it has one node from each "clause triangle." Set those terms to **true**. This is possible because no 2 are negations of each other. $\square$

3-Dimensional Matching is NP-complete

Two-Dimensional Matching



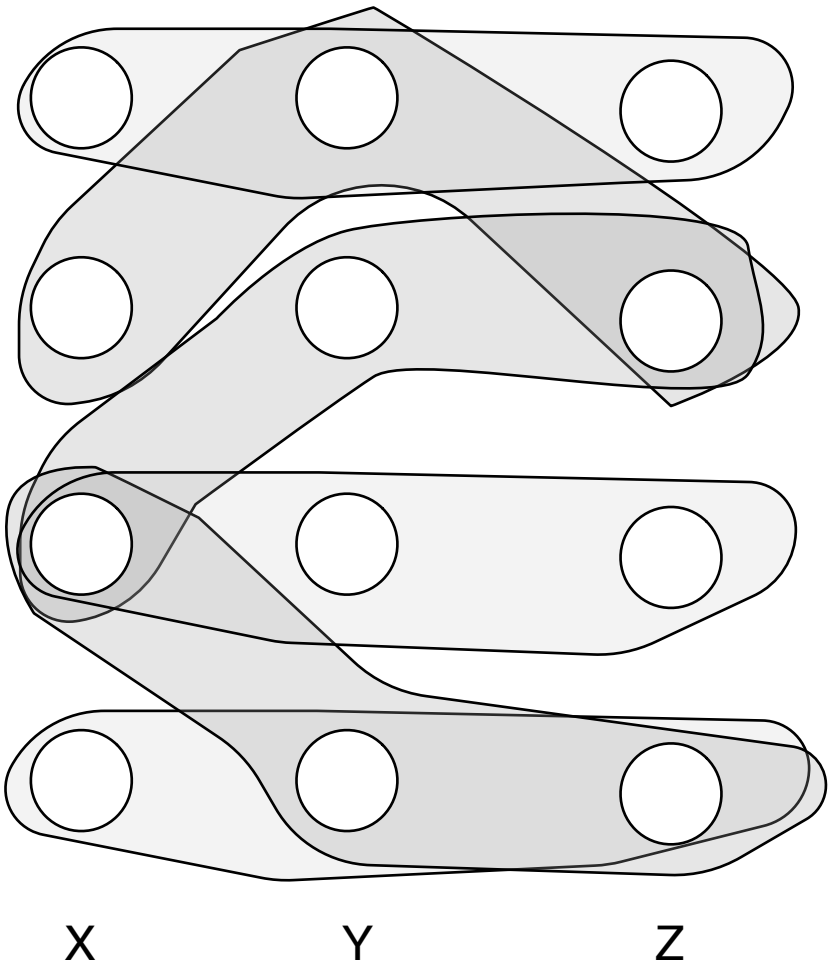
Recall '2-d matching':

Given sets X and Y, each with n elements, and a set E of pairs $\{x, y\}$,

Question: is there a choice of pairs such that every element in $X \cup Y$ is paired with some other element?

Usually, we thought of edges instead of pairs: $\{x, y\}$, but they are really the same thing.

Three-Dimensional Matching



Given: Sets X, Y, Z, each of size n, and a set $T \subset X \times Y \times Z$ of order triplets.

Question: is there a set of n triplets in T such that each element is contained in exactly one triplet?

3DM Is NP-Complete

Theorem. *Three-dimensional matching (aka 3DM) is NP-complete*

Proof. 3DM is in NP: a collection of n sets that cover every element exactly once is a certificate that can be checked in polynomial time.

Reduction from 3-SAT. We show that:

$$\text{3-SAT} \leq_P \text{3DM}$$

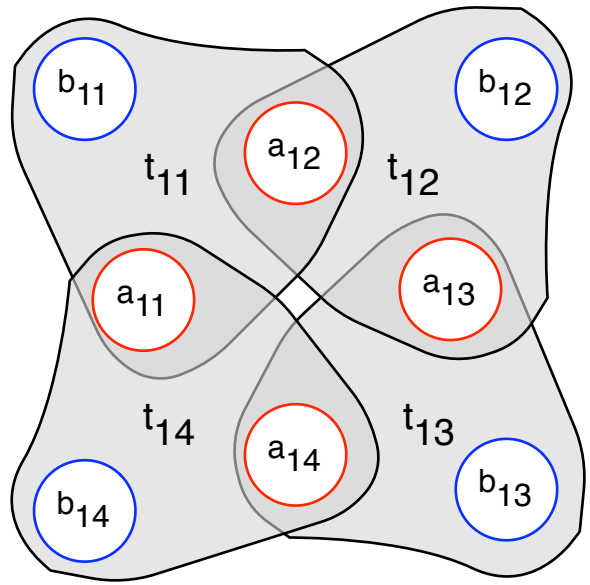
In other words, if we could solve 3DM, we could solve 3-SAT.

3-SAT $\leq_P$ 3DM

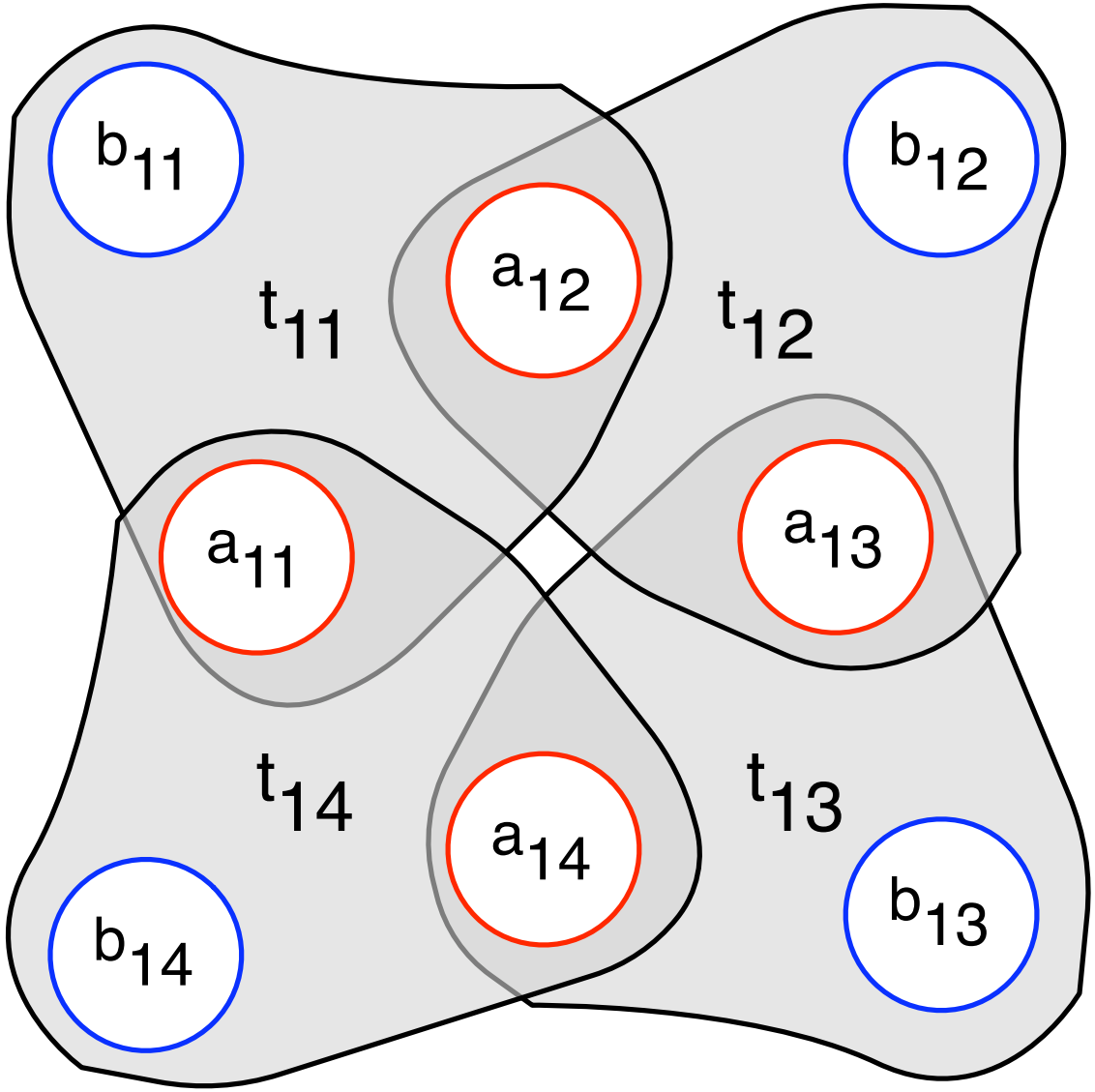
3SAT instance: $x_1, \ldots, x_n$ be n boolean variables, and $C_1, \ldots, C_k$ clauses.

We create a gadget for each variable x_i:

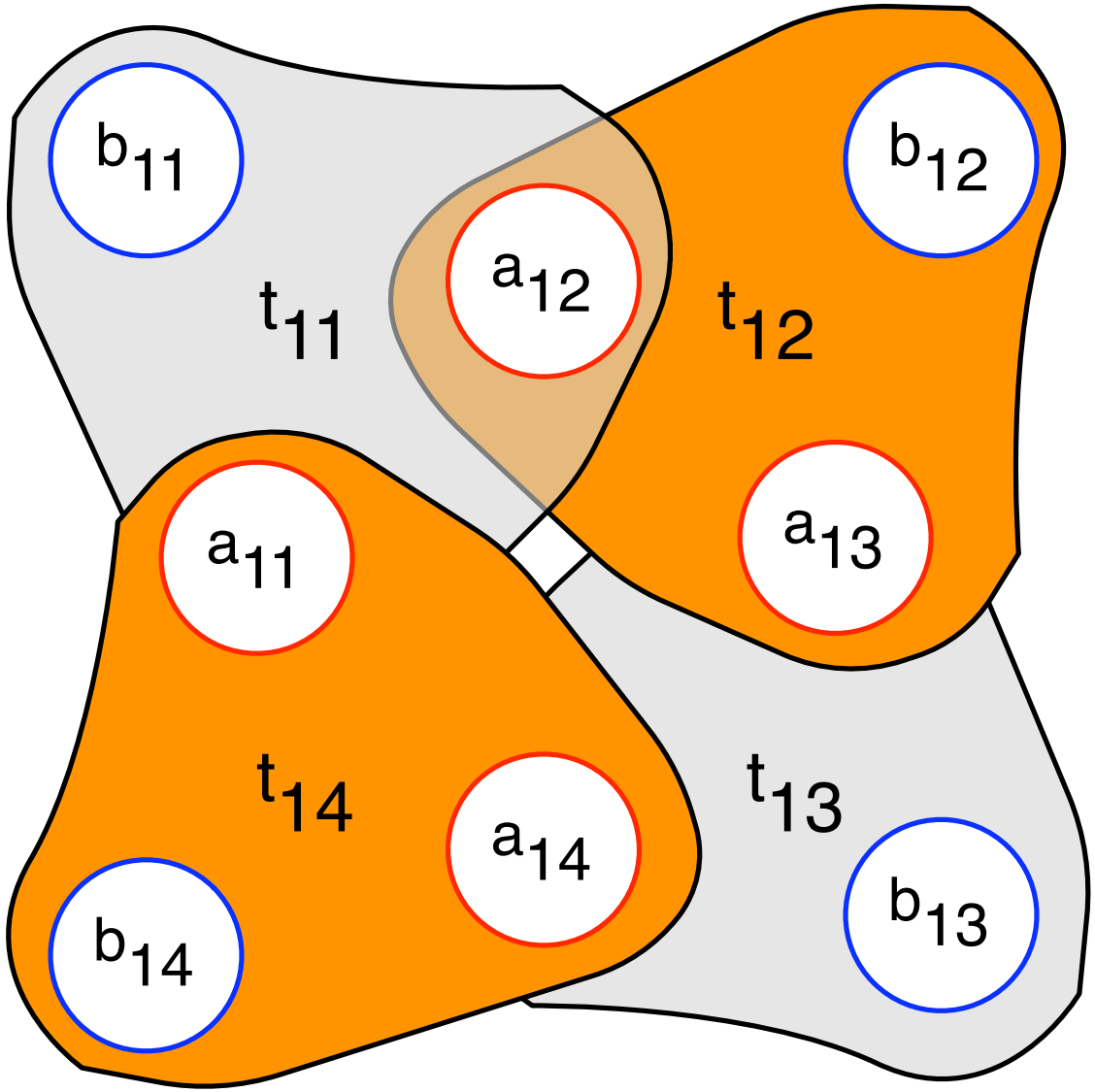
$$A_i = \{a_{i1}, \ldots, a_{i,2k}\} \quad \textit{core}$$
$$B_i = \{a_{i1}, \ldots, a_{i,2k}\} \quad \textit{tips}$$
$$t_{ij} = (a_{ij}, a_{i,j+1}, b_{ij}) \quad \textit{TF triples}$$



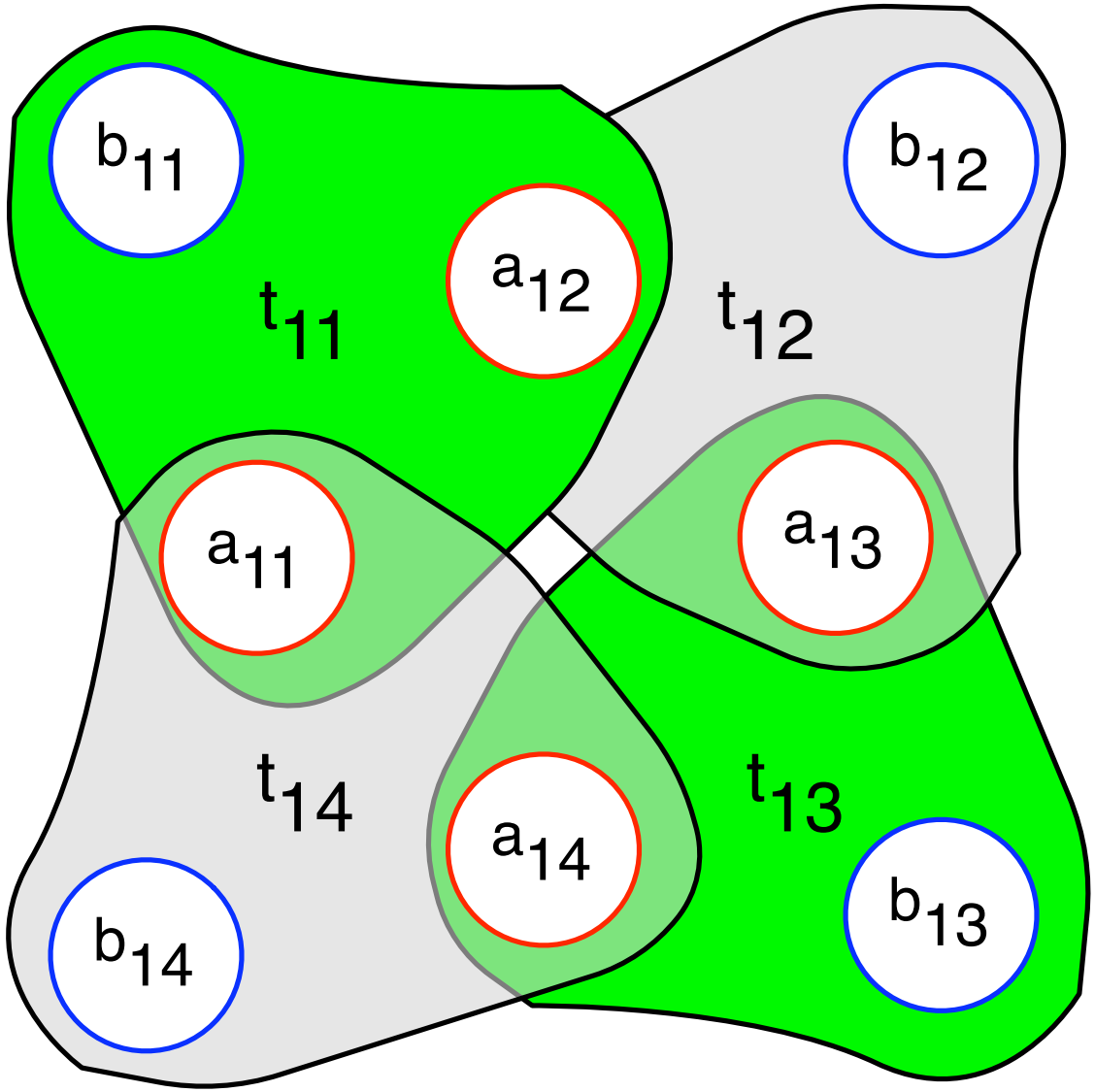
Gadget Encodes True and False



Gadget Encodes True and False

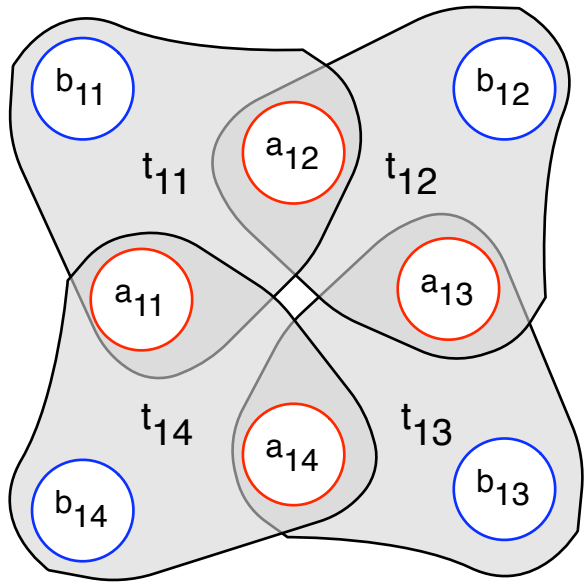


Gadget Encodes True and False



How "choice" is encoded

- We can only either use the even or odd "wings" of the gadget.
- In other words, if we use the even wings, we leave the odd tips uncovered (and vice versa).
- Leaving the odd tips free for gadget i means setting x_i to **false**.
- Leaving the odd tips free for gadget i means setting x_i to **true**.



Clause Gadgets

Need to encode constraints between the tips that ensure we satisfy all the clauses.

We create a gadget for each clause $C_j = \{t_1, t_2, t_3\}$

$$P_j = \{c_j, c'_j\} \quad \textit{Clause core}$$

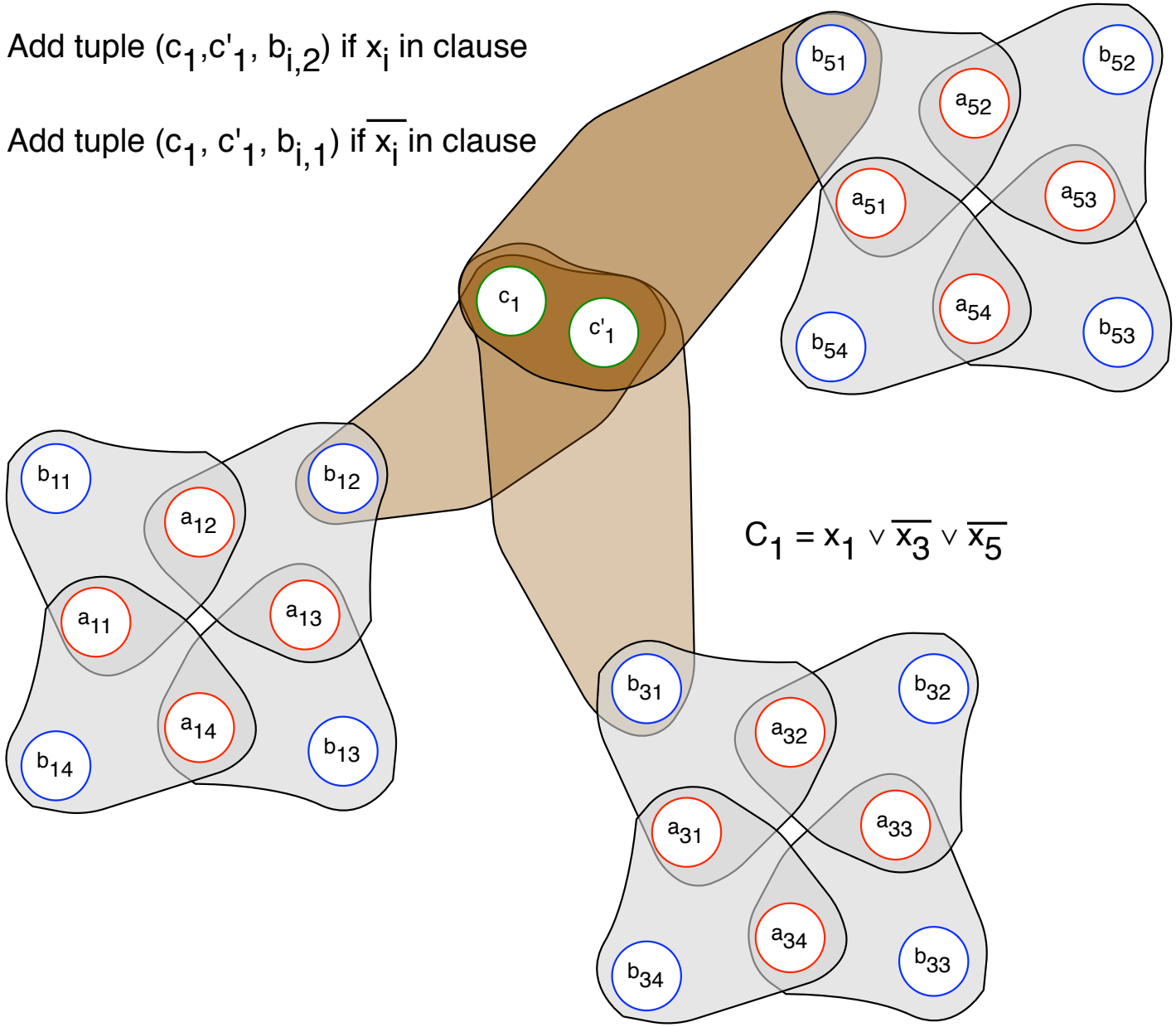
We will hook up these two clause core nodes with some tip nodes depending on whether the clause asks for a variable to be true or false.

See the next slide.

Clause Gadget Hookup

Add tuple $(c_1, c'_1, b_{i,2})$ if x_i in clause

Add tuple $(c_1, c'_1, b_{i,1})$ if $\overline{x_i}$ in clause



$C_1 = x_1 \vee \overline{x_3} \vee \overline{x_5}$

Clause Gadgets

Since only clause tuples (brown) cover c_j, c'_j, we have to choose exactly one of them for every clause.

We can only choose a clause tuple (c_j, c'_j, b_{ij}) if we haven't chosen a TF tuple that already covers b_{ij}.

Hence, we can satisfy (cover) the clause (c_j, c'_j) with the term represented by b_{ij} only if we "set" x_i to the appropriate value.

That's the basic idea. Two technical points left...

Details

Need to cover all the tips:

Even if we satisfy all the clauses, we might have extra tips left over. We add a clean up gadget (q_i, q_i', b) for every tip b.

Can we partition the sets?

$$X = \{a_{ij} : j \text{ even}\} \cup \{c_j\} \cup \{q_i\}$$
$$Y = \{a_{ij} : j \text{ odd}\} \cup \{c_j'\} \cup \{q_i'\}$$
$$Z = \{b_{ij}\}$$

Every set we defined uses 1 element from each of X, Y, Z.

Proof

If there is a satisfying assignment,

We choose the odd / even wings depending on whether we set a variable to **true** or **false**. At least 1 free tip for a term will be available to use to cover each clause gadget. We then use the clean up gadgets to cover all the rest of the tips.

If there is a 3D matching,

We can set variable x_i to **true** or **false** depending on whether it's even or odd wings were chosen. Because $\{c_j, c'_j\}$ were covered, we must have correctly chosen one even/odd wing that will satisfy this clause.