# 15-451/651 Algorithms, Fall 2017
# Recitation #9 Worksheet

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**Rock-Paper-Scissors with a Twist** Suppose we have a non-standard game of rock-paper-scissors, which is still zero-sum, but with the following payoffs for the row player (Alice):

Bob plays

| Alice plays | $r$ | $p$ | $s$ |
|---|---|---|---|
| $r$ | 0 | $-1$ | 2 |
| $p$ | 1 | 0.5 | $-1$ |
| $s$ | $-1$ | 2 | $-1$ |

1. If Alice decides to play $\mathbf{p} = (p_1, p_2, 1 - p_1 - p_2)$ as her strategy, what should Bob play to minimize the payoff to Alice? What is Alice's payoff if he does this.

2. Hence, write down the linear program that Alice must solve in order to find her best strategy $\mathbf{p}^\star$. You don't have to solve for the value of this game.

**Coding up shortest paths as an LP:** You can code up the $s$-$t$ shortest-path problem as an LP. The input is a directed graph $G$ with edge weights $w(e) \geq 0$, start node $s$, and a target $t$. We want to find a path from $s$ to $t$ of least weight.

1. *What are the variables?* Suppose we have a variable $d_v$ for every vertex $v$, representing its distance from $s$. What are the constraints you should write?

2. The objective? We claim it is *maximize* $d_t$. Why does this make sense?

**Coding up shortest paths as an LP (Approach II):** Have a variable $x_e$ for each edge $e$, with constraints that $0 \leq x_e \leq 1$. (Think of $x_e = 1$ meaning we use that edge, and $x_e = 0$ meaning we don't, but of course the LP might assign fractional values.) Our goal is to minimize $\sum_e w(e) x_e$, subject to:

- One unit of "flow" leaves $s$: $\sum_{e=(s,v)} x_{sv} = 1$
- One unit of "flow" enters $t$: $\sum_{e=(v,t)} x_{vt} = 1$.
- For all $v \notin \{s, t\}$, we have flow-in = flow-out: $\sum_{e=(u,v)} x_{uv} = \sum_{e=(v,u)} x_{vu}$.

This is a min-cost flow (send 1 unit of $s$-$t$-flow with least cost). You can put edge-capacity 1, but since we send only 1 unit of flow, the capacity constraints don't matter.

1. Solve the LP to get an optimal solution. If we get back an integer solution (i.e., if $x_e \in \{0, 1\}$ for each edge $e$) argue that the edges with $x_e = 1$ give a shortest $s$-$t$ path.

2. Suppose the optimal LP solution returns a fractional flow. Argue that any flow-carrying path from $s$ to $t$ is a shortest path.

**Taking Duals** Consider this maximization linear program:

$$\begin{aligned} &\max(x_1 + 3x_2 - 3x_3) \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 \le 2 \\ & 7x_1 + 2x_2 + 5x_3 \le 6 \\ & 2x_1 + 2x_2 - x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

1. Write down its dual LP. (Is it a maximization or minimization problem? What are the variables? Constraints?)

2. What is the dual of the dual?

**Taking Duals II** Write down the dual of this minimization LP. Some of the inequalities are greater-than and some are less-than, and not all contraints have all variables

$$\begin{aligned} &\max(x_1 - 3x_2 + 2x_3) \\ \text{s.t.} \quad & 3x_1 + 2x_2 \ge 2 \\ & 2x_2 - x_3 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

**Minimax from Duality** Let the row-player's payoff be given by this (non-negative) matrix

| | $L$ | $R$ |
|---|---|---|
| $L$ | 1 | 5 |
| $R$ | 3 | 2 |

1. If the probabilities on the two rows are $p_1$ and $p_2$, write down an LP for the row player's optimal strategy.

2. Now take the dual of this LP. Show thhis dual is an LP computing the column player's optimal strategy (And hence string duality implies the minimax theorem)