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Proofs for the assumptions we made in the recitation problem.
The game matrix looks like this (problem statement is in the recitation note):

	search 1	search 2	search 3	
p_1	3	1	2	balanced tree
p_2	1	3	2	zag-zig
p_3	2	3	1	zig-zag
p_4	1	2	3	zag-zag
p_5	3	2	1	zig-zig

Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ be a randomized strategy. ($0 \le p_i \le 1$ and $\sum p_i = 1$)

Claim 1: We can safely assume that the optimal strategy have $p_2 = p_3$ and $p_4 = p_5$.

We will prove the claim by proving more general statement that implies Claim 1.

More general statement: For any randomized strategy $P = \{p_1, p_2, p_3, p_4, p_5\}$, we can create a better (or equally good) strategy by having $P' = \{p'_1, p'_2, p'_3, p'_4, p'_5\}$ where $p'_1 = p_1$, $p'_2 = p'_3 = (p_2 + p_3)/2$ and $p'_4 = p'_5 = (p_4 + p_5)/2$.

Proof:

First note that P' is an valid strategy because $0 \le p'_i \le 1$ and $\sum p'_i = \sum p_i = 1$.

Let E_1, E_2 and E_3 be the expected cost of strategy P if input is 1,2 and 3 repectively. Then,

$E_1 = 2p_1 + p_2 + 2p_3 + p_4 + 3p_5$
$E_2 = p_1 + 3p_2 + 3p_3 + 2p_4 + 2p_5$
$E_3 = 2p_1 + 2p_2 + p_3 + 3p_4 + p_5$

Then the expected worst case cost for strategy P is $MAX(E_1, E_2, E_3)$

Similarly, let E'_1, E'_2 and E'_3 be the expected cost of strategy P' if input is 1,2 and 3 repectively. Then,

$E'_1 = 2p'_1 + p'_2 + 2p'_3 + p'_4 + 3p'_5 = 2p_1 + 3/2(p_2 + p_3) + 2(p_4 + p_5)$
$E'_2 = p'_1 + 3p'_2 + 3p'_3 + 2p'_4 + 2p'_5 = p_1 + 3p_2 + 3p_3 + 2p_4 + 2p_5 = E_2$
$E'_3 = 2p'_1 + 2p'_2 + p'_3 + 3p'_4 + p'_5 = 2p_1 + 3/2(p_2 + p_3) + 2(p_4 + p_5) = E'_1$

Then the expected worst case cost for strategy P' is $MAX(E'_1, E'_2, E'_3)$

We want to show $MAX(E'_1, E'_2, E'_3) \le MAX(E_1, E_2, E_3)$

Case 1: $MAX(E_1, E_2, E_3) = E_1$
That means $E_1 - E_3 \ge 0 \Rightarrow -p_2 + p_3 - 2p_4 + 2p_5 \ge 0$.
Consider $E_1 - E'_1 = -1/2p_2 + 1/2p_3 - p_4 + p_5 = 1/2(E_1 - E_3) \ge 0$ Thus $E_1 \ge E'_1$. We know $E'_2 = E_2 \le E_1$ and $E'_3 = E'_1 \le E_1$. All E'_1, E'_2 and $E'_3 \le E_1 = MAX(E_1, E_2, E_3)$. Thus $MAX(E'_1, E'_2, E'_3) \le MAX(E_1, E_2, E_3)$

Case 2: $MAX(E_1, E_2, E_3) = E_2$
That means $E_2 - E_1 = -p_1 + 2p_2 + p_3 + p_4 - p_5 \ge 0$ and $E_2 - E_3 = -p_1 + p_2 + 2p_3 - p_4 + p_5 \ge 0$.
Then $(E_2 - E_1) + (E_2 - E_3) = -2p_1 + 3p_2 + 3p_3 \ge 0$.
Consider $E'_2 - E'_1 = -p_1 + 3/2p_2 + 3/2p_3 = ((E_2 - E_1) + (E_2 - E_3))/2 \ge 0$.
Thus $E'_2 \ge E'_1 = E'_3 \Rightarrow MAX(E'_1, E'_2, E'_3) = E'_2 = E_2 = MAX(E_1, E_2, E_3)$.

Case 3: $MAX(E_1, E_2, E_3) = E_3$
Analogous with Case 1: $MAX(E'_1, E'_2, E'_3) \le MAX(E_1, E_2, E_3)$

In all cases, we have $MAX(E'_1, E'_2, E'_3) \le MAX(E_1, E_2, E_3)$ ■

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Claim 2: We can safely assume that the optimal strategy have $p_4 = p_5 = 0$ (Do not need to use zag-zag and zig-zig trees).

Proof:

Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ is an optimal strategy. By Claim 1, we can assume $p_2 = p_3$ and $p_4 = p_5$. Let $q_1 = p_1$, $q_{23} = p_2 = p_3$, $q_{45} = p_4 = p_5$. Then $(q_1 + 2q_{23} + 2q_{45} = 1)$

Let E_1, E_2 and E_3 be the expected cost of strategy P if input is 1,2 and 3 repectively. Then,

$E_1 = 2p_1 + p_2 + 2p_3 + p_4 + 3p_5 = 2q_1 + 3q_{23} + 4q_{45}$
$E_2 = p_1 + 3p_2 + 3p_3 + 2p_4 + 2p_5 = q_1 + 6q_{23} + 4q_{45}$
$E_3 = 2p_1 + 2p_2 + p_3 + 3p_4 + p_5 = 2q_1 + 3q_{23} + 4q_{45}$

Consider a new strategy $P' = \{p'_1, p'_2, p'_3, p'_4, p'_5\}$ where $p'_1 = p_1 + p_4 + p_5 = q_1 + 2q_{45}$, $p'_2 = p'_3 = p_2 = p_3 = q_{23}$ and $p'_4 = p'_5 = 0$. It is a vaild strategy as $0 \le p'_i \le 1$ and $\sum p'_i = \sum p_i = 1$. Then the expected costs for P' is

$E'_1 = 2p'_1 + p'_2 + 2p'_3 = 2q_1 + 4q_{45} + 3q_{23}$
$E'_2 = p'_1 + 3p'_2 + 3p'_3 = q_1 + 2q_{45} + 6q_{23}$
$E'_3 = 2p'_1 + 2p'_2 + p'_3 = 2q_1 + 4q_{45} + 3q_{23}$

Clearly, $E'_1 = E_1$, $E'_2 \le E_2$ and $E'_3 = E_3$.
Thus $MAX(E'_1, E'_2, E'_3) \le MAX(E_1, E_2, E_3)$.

So we created a better (or equally good) strategy that does not use zag-zag and zig-zig trees at all.
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