Proofs for the assumptions we made in the recitation problem. 
The game matrix looks like this (problem statement is in the recitation note):

<table>
<thead>
<tr>
<th></th>
<th>search 1</th>
<th>search 2</th>
<th>search 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$p_2$</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$p_3$</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$p_4$</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$p_5$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

$E$ is a balanced tree.

Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ be a randomized strategy. $(0 \leq p_i \leq 1$ and $\sum p_i = 1)$

**Claim 1:** We can safely assume that the optimal strategy have $p_2 = p_3$ and $p_4 = p_5$.

We will prove the claim by proving more general statement that implies Claim 1.

**More general statement:** For any randomized strategy $P = \{p_1, p_2, p_3, p_4, p_5\}$, we can create a better (or equally good) strategy by having $P' = \{p_1', p_2', p_3', p_4', p_5'\}$ where $p_1' = p_1$, $p_2' = p_3' = (p_2 + p_3)/2$ and $p_4' = p_5' = (p_4 + p_5)/2$.

**Proof:**

First note that $P'$ is an valid strategy because $0 \leq p_i' \leq 1$ and $\sum p_i' = \sum p_i = 1$.

Let $E_1$, $E_2$ and $E_3$ be the expected cost of strategy $P$ if input is 1,2 and 3 respectively. Then,

$E_1 = 2p_1 + p_2 + 2p_3 + p_4 + 3p_5$
$E_2 = p_1 + 3p_2 + 3p_3 + 2p_4 + 2p_5$
$E_3 = 2p_1 + 2p_2 + p_3 + 3p_4 + p_5$

Then the expected worst case cost for strategy $P$ is $MAX(E_1, E_2, E_3)$

Similarly, let $E_1'$, $E_2'$ and $E_3'$ be the expected cost of strategy $P'$ if input is 1,2 and 3 respectively. Then,

$E_1' = 2p_1' + p_2' + 3p_3' + 2p_4' + 3p_5' = 2p_1 + 3/2(p_2 + p_3) + 2(p_4 + p_5)$
$E_2' = p_1' + 3p_2' + 3p_3' + 2p_4' + 2p_5' = p_1 + 3/2(p_3 + 2p_4 + 2p_5) = E_2$
$E_3' = 2p_1' + 2p_2' + p_3' + 3p_4' + p_5' = 2p_1 + 3/2(p_2 + p_3) + 2(p_4 + p_5) = E_1'$

Then the expected worst case cost for strategy $P'$ is $MAX(E_1', E_2', E_3')$

We want to show $MAX(E_1', E_2', E_3') \leq MAX(E_1, E_2, E_3)$

**Case 1:** $MAX(E_1, E_2, E_3) = E_1$

That means $E_1 - E_3 \geq 0 \Rightarrow -p_2 + p_3 - 2p_4 + 2p_5 \geq 0$.

Consider $E_1 - E_1' = -1/2p_2 + 1/2p_3 - p_4 + p_5 = 1/2(E_1 - E_3) \geq 0$ Thus $E_1 \geq E_1'$. We know $E_2' = E_2 \leq E_1$ and $E_3' = E_1' \leq E_1$. All $E_1'$, $E_2'$ and $E_3'$ $E_1 = MAX(E_1, E_2, E_3)$. Thus $MAX(E_1', E_2', E_3') \leq MAX(E_1, E_2, E_3)$

**Case 2:** $MAX(E_1, E_2, E_3) = E_2$

That means $E_2 - E_3 = -p_1 + 2p_2 + 3p_3 - p_4 + p_5 \geq 0$ and $E_2 - E_3 = -p_1 + p_2 + 2p_3 - 4p_4 + p_5 \geq 0$.

Then $(E_2 - E_1) + (E_2 - E_3) = -2p_1 + 3p_2 + 3p_3 \geq 0$.

Consider $E_2 - E_1' = -p_1 + 2p_2 + 3p_3 = ((E_2 - E_1) + (E_2 - E_3))/2 \geq 0$.

Thus $E_2' \geq E_1'$. Thus $E_1' = E_1 = MAX(E_1', E_2', E_3') = E_2 = E_2 = MAX(E_1, E_2, E_3)$.

**Case 3:** $MAX(E_1, E_2, E_3) = E_3$

Analogous with Case 1: $MAX(E_1', E_2', E_3') \leq MAX(E_1, E_2, E_3)$

In all cases, we have $MAX(E_1', E_2', E_3') \leq MAX(E_1, E_2, E_3)$
Claim 2: We can safely assume that the optimal strategy have $p_4 = p_5 = 0$ (Do not need to use zag-zag and zig-zig trees).

Proof:

Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ is an optimal strategy. By Claim 1, we can assume $p_2 = p_3$ and $p_4 = p_5$.

Let $q_1 = p_1$, $q_{23} = p_2 = p_3$, $q_{45} = p_4 = p_5$. Then $(q_1 + 2q_{23} + 2q_{45} = 1)$

Let $E_1, E_2$ and $E_3$ be the expected cost of strategy $P$ if input is 1, 2 and 3 respectively. Then,

$$E_1 = 2p_1 + p_2 + 2p_3 + p_4 + 3p_5 = 2q_1 + 3q_{23} + 4q_{45}$$
$$E_2 = p_1 + 3p_2 + 3p_3 + 2p_4 + 2p_5 = q_1 + 6q_{23} + 4q_{45}$$
$$E_3 = 2p_1 + 2p_2 + p_3 + 3p_4 + p_5 = 2q_1 + 3q_{23} + 4q_{45}$$

Consider a new strategy $P' = \{p'_1, p'_2, p'_3, p'_4, p'_5\}$ where $p'_1 = p_1 + p_4 + p_5 = q_1 + 2q_{45}$, $p'_2 = p'_3 = p_2 = p_3 = q_{23}$ and $p'_4 = p'_5 = 0$. It is a valid strategy as $0 \leq p'_i \leq 1$ and $\sum p'_i = \sum p_i = 1$. Then the expected costs for $P'$ is

$$E'_1 = 2p'_1 + p'_2 + 2p'_3 = 2q_1 + 4q_{45} + 3q_{23}$$
$$E'_2 = p'_1 + 3p'_2 + 3p'_3 = q_1 + 2q_{45} + 6q_{23}$$
$$E'_3 = 2p'_1 + 2p'_2 + p'_3 = 2q_1 + 4q_{45} + 3q_{23}$$

Clearly, $E'_1 = E_1$, $E'_2 \leq E_2$ and $E'_3 = E_3$.

Thus $\text{MAX}(E'_1, E'_2, E'_3) \leq \text{MAX}(E_1, E_2, E_3)$.

So we created a better (or equally good) strategy that does not use zag-zag and zig-zig trees at all. ■