Claim: Suppose $T(n) = n^b + T(a_1 n) + T(a_2 n) + ... + T(a_k n)$ where $a1^b + ... + ak^b < 1$ (with constant base case such as $T(x) = 0(\forall x \leq 5)$). Then $T(n) = O(n^b)$.

Proof:

We want to show $(\exists c > 0)(\exists n_0 > 0)s.t.(\forall n \geq n_0)T(n) \leq cn^b$.

Let $a1^b + ... + ak^b = A$. Then we have $A < 1$. Set $c = 2/(1 - A)$ and $n_0 = 5$. Note that $c = 2/(1 - A) > 0$ as $A < 1$. We can inductively prove $(\forall n \geq 1)T(n) \leq cn^b$.

Base case is trivial by the base case of the recurrence.
Suppose $T(K) < cK^b$ for all $K < N$ –induction hypothesis.
Want to show that it holds for N as well.

$T(N) = N^b + T(a_1 N) + T(a_2 N) + ... + T(a_k N)$
$\leq N^b + c(a_1 N)^b + c(a_2 N)^b + ... + c(a_k N)^b$ by induction hypothesis
$= N^b + cN^b(a_1^b + a_2^b + ... + a_k^b)$
$= N^b + cN^b A$
$= N^b + (2/(1 - A))AN^b$ plugging in c.
$= N^b(1 + \frac{2A}{1-A})$
$= N^b(\frac{1+A}{1-A})$
$< N^b(\frac{2}{1-A})$ as $A < 1$
$= cN^b$ ■

Of course, magically guessing c is hard. So try induction steps with unknown c, then it becomes easier to figure out what c should look like. One of the key things of this proof is that we need to utilize $1 - A > 0$ to choose a proper c (where $a1^b + ... + ak^b = A$).