Locality - 2

15-411/15-611 Compiler Design

Seth Copen Goldstein

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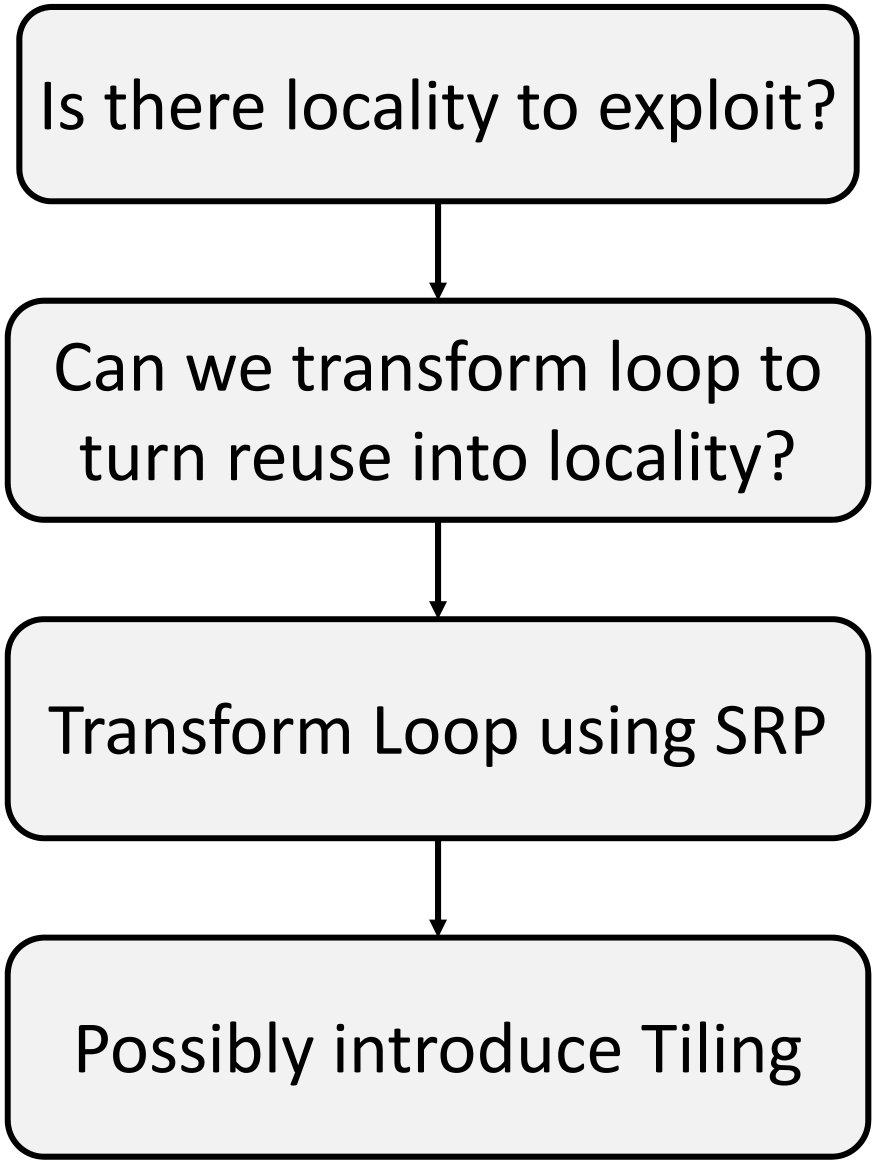
Our Goal: Increase locality


Is there locality to exploit?
Can we transform loop to turn reuse into locality?
Transform Loop using SRP
Possibly introduce Tiling

Use Reuse Analysis to determine amount of possible reuse.

Use dependence information to determine pace of possible transformations.

Perform unimodular transformations.

turn n-deep into 2n-deep

Our Goal: Increase locality

Is there locality to exploit?

Use Reuse Analysis to determine amount of possible reuse

Key idea: Treat each iteration of loop nest as a point in an n-dimentional iteration space.

transformations.

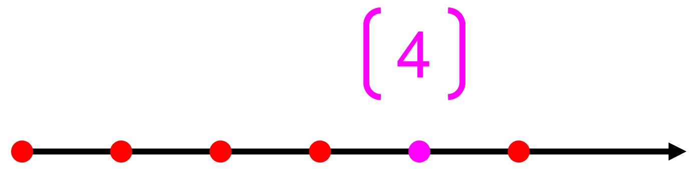
Possibly introduce Tiling

turn n-deep into 2n-deep

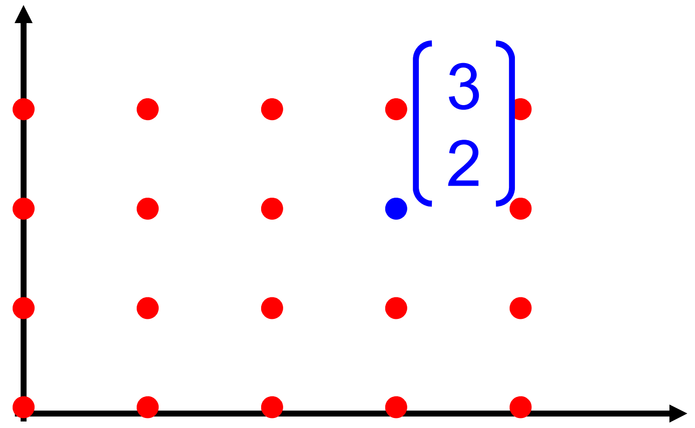
Iteration Space

Every iteration generates a point in an n-dimensional space, where n is the depth of the loop nest.

```
for (i=0; i<n; i++) {
   •••
}
```



```
for (i=0; i<n; i++)
   for (j=0; j<4; j++) {
      •••
}
```



Iteration Vectors

- Given a nest of n loops, the iteration vector *i* of a particular iteration of the innermost loop is a vector of integers that contains the iteration numbers for each of the loops in order of nesting level.
- Thus, the iteration vector is: $\{i_1, i_2, ..., i_n\}$ where i_k, $1 \leq k \leq n$ represents the iteration number for the loop at nesting level k

Ordering of Iteration Vectors

- An ordering for iteration vectors
- Use an intuitive, lexicographic order
- Iteration i precedes iteration j, denoted $i < j$, iff:
    1. $i[1:n-1] < j[1:n-1]$, or
    2. $i[1:k-1] = j[1:k-1]$ and $i_k < j_k$

$$\begin{pmatrix} i_1 \\ i_2 \\ \dots \\ i_k \\ \dots \\ i_n \end{pmatrix} < \begin{pmatrix} j_1 \\ j_2 \\ \dots \\ j_k \\ \dots \\ j_n \end{pmatrix}$$

Uniformly Generated references

- f and g are indexing functions: $Z^n \rightarrow Z^d$
  - n is depth of loop nest
  - d is dimensions of array, A
- Two references A[f($\mathbf{i}$)] and A[g($\mathbf{i}$)] are uniformly generated if

$$f(\mathbf{i}) = H\mathbf{i} + c_f \text{ AND } g(\mathbf{i})=H\mathbf{i}+c_g$$

- H is a linear transform
- c_f and c_g are constant vectors

Uniformly generated sets

```
for $I_1$ := 0 to 5
  for $I_2$ := 0 to 6
    A[$I_2$+1] = 1/3 * (A[$I_2$]+A[$I_2$+1]+A[$I_2$+2])
```

A[I_2+1] $\quad [\,0\ 1\,]\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + [\,1\,]$

A[I_2] $\quad [\,0\ 1\,]\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + [\,0\,]$

A[I_2+2] $\quad [\,0\ 1\,]\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + [\,2\,]$

Predicting Cache Behavior through "Locality Analysis"

- Definitions:
  - Reuse:
    accessing a location that has been accessed in the past
  - Locality:
    accessing a location that is now found in the cache
- Key Insights
  - Locality only occurs when there is reuse!
  - BUT, reuse does not necessarily result in locality.
  - Why not?

Steps in Locality Analysis

1. Find data reuse
   - if caches were infinitely large, we would be finished
2. Determine “localized iteration space”
   - set of inner loops where the data accessed by an iteration is expected to fit within the cache
3. Find data locality:
   - reuse $\supseteq$ localized iteration space $\supseteq$ locality

Self-Temporal

- For a reference, A[H**i**+**c**], there is self-temporal reuse between **m** and **n** when H**m**+**c**=H**n**+**c**, i.e., H(**r**)=**0**, where **r**=**m**-**n**.
- The direction of reuse is **r**.
- The self-temporal reuse vector space is: R_{ST} = Ker H
- Amount of reuse is $s^{\dim(Rst)}$
- There is locality if $R_{ST} \subseteq$ localized vector space.
- $R_{ST} \cap L$ = locality
- # of mem refs = $\frac{1}{s^{\dim(R_{ST} \cap L)}}$

Examples of reuse

```
for $I_1$ := 0 to 5
  for $I_2$ := 0 to 6
    A[$I_2$ + 1] = 1/3 * (A[$I_2$]+ A[$I_2$ + 1] + A[$I_2$ + 2])
```

Uniformly Generated Set:

{A[I_2], A[I_2+1] ,A[I_2+2]} H = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>Type</u>	<u>reuse space</u>	<u>reuse factor</u>
Self-Temporal:	Ker(H) = span{(1,0)}	s

Self-Spatial

- Occurs when we access in order
- Spatial reuse occurs when only last index varies
- So, all but last row of H must be identical
- H_s := H with last row set to 0
- self-spatial reuse vector space = R_{SS}

  $R_{SS} = \ker H_S$
- Notice, $\ker H \subseteq \ker H_s$
- If, $R_{ss} \cap L = R_{ST} \cap L$, then no additional benefit to self-spatial reuse
- $\frac{k}{k^{\dim(R_{ST} \cap L)}}$ memory accesses/iteration

Examples of reuse

```
for $I_1$ := 0 to 5
  for $I_2$ := 0 to 6
    A[$I_2$ + 1] = 1/3 * (A[$I_2$]+ A[$I_2$ + 1] + A[$I_2$ + 2])
```

Uniformly Generated Set:

{A[I_2], A[I_2+1] ,A[I_2+2]} $H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $H_s = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Type	reuse space	reuse factor
Self-Temporal:	Ker(H) = span{(1,0)}	s
Self-Spatial:	Ker(H_s) = span{(1,0),(0,1)}	L

Group Reuse

- Occurs between **different** references in a loop nest when they access
  - the same element in the reuse vector space
  - the same cache line in the reuse vector space

Examples of reuse

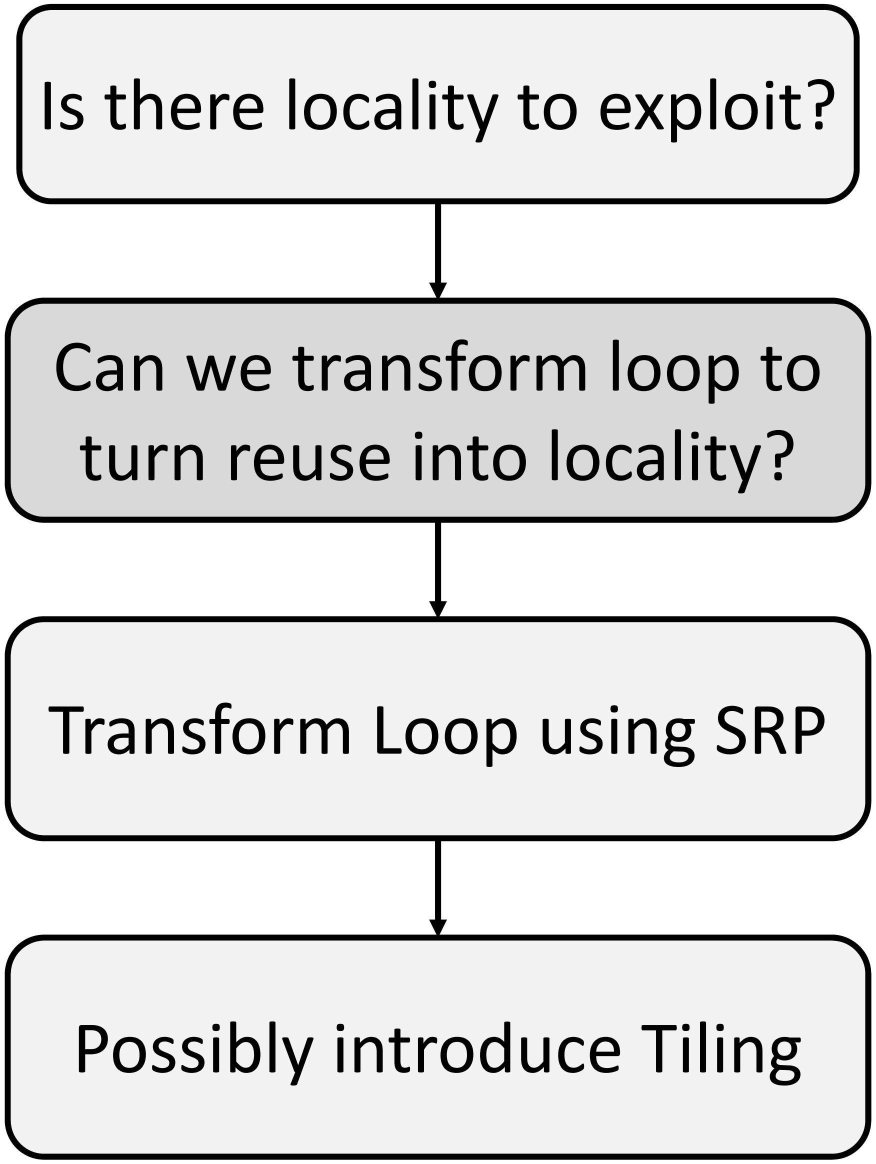
```
for $I_1$ := 0 to 5
  for $I_2$ := 0 to 6
    A[$I_2$ + 1] = 1/3 * (A[$I_2$]+ A[$I_2$ + 1] + A[$I_2$ + 2])
```

Uniformly Generated Set:

{A[I_2] , A[I_2+1] ,A[I_2+2]} H = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Type	reuse space	reuse factor
Self-Temporal:	Ker(H) = span{(1,0)}	s
Self-Spatial:	Ker(H_s) = span{(1,0),(0,1)}	L
Group-Temporal:	span{(1,0),(0,1)}	3

Our Goal: Increase locality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Transform Loop using SRP
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Use Reuse Analysis to determine amount of possible reuse.

Use dependence information to determine pace of possible transformations.

Perform unimodular transformations.

turn n-deep into 2n-deep

Loop Dependence

- There exists a dependence from statements S_1 to statement S_2 in a common nest of loops iff there exist two iteration vectors *i* and *j* for the nest, st.

(1) (a) $i < j$ or Loop Carried

(b) $i = j$ and there is a path from S_1 to S_2 in the body of the loop, Loop independent

(2) statement S_1 accesses memory location *M* on iteration *i* and statement S_2 accesses location *M* on iteration *j*, and

(3) one of these accesses is a write.

Dependence Distance

- Using iteration vectors and def of dependence we can determine the distance of a dependence:
- In n-deep loop nest if
  - S1 is source in iteration i
  - S2 is sink in iteration j
- Distance of dependence is represented with a distance vector: D
  - Vector of length n, where
  - $d_k = j_k - i_k$

Distance Vector

```
for (i=0; i<n; i++) {
    A[i] = B[i];
    B[i+1] = A[i];
}
```

```
A[0] = B[0];  }
B[1] = A[0];  }  i=0
A[1] = B[1];  }
B[2] = A[1];  }  i=1
A[2] = B[2];  }
B[3] = A[2];  }  i=2
      ⋮
```

Distance vector is the difference between the target and source iterations.

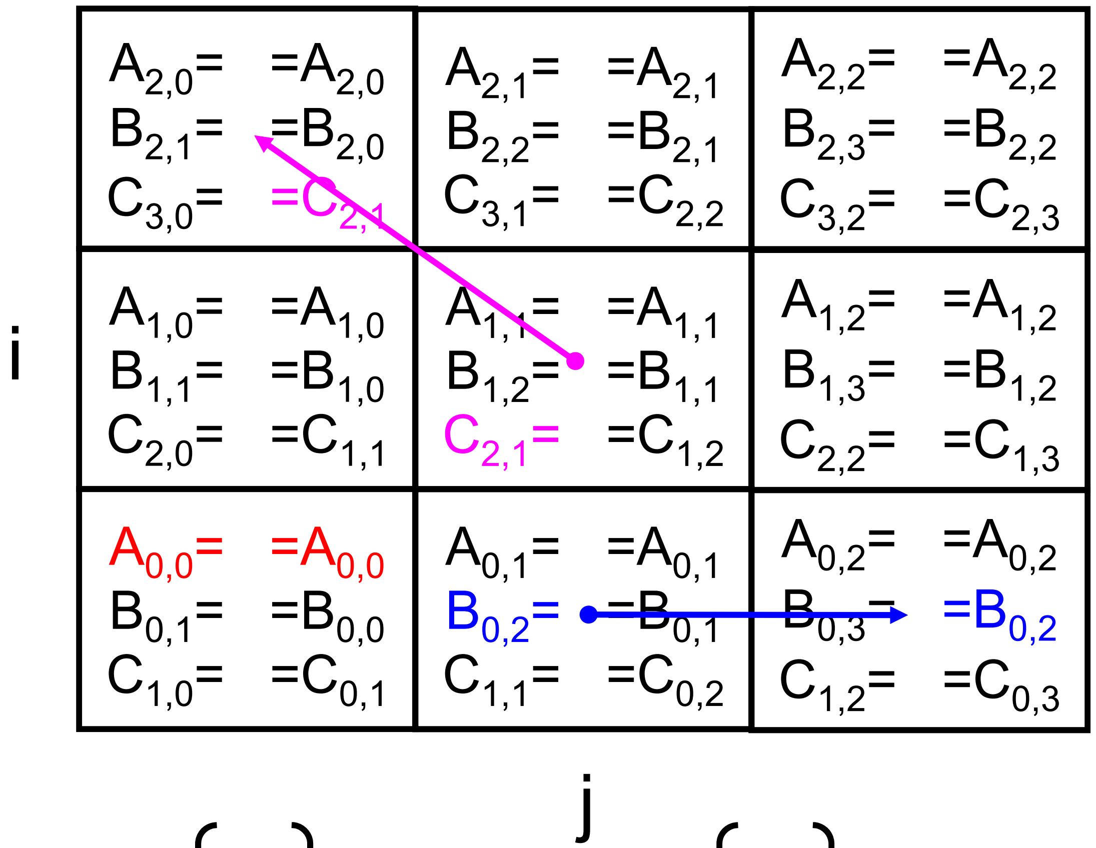
$$\mathbf{d = I_t - I_s}$$

Exactly the distance of the dependence, i.e.,

$$\mathbf{I_s + d = I_t}$$

Example of Distance Vectors

```
for (i=0; i<n; i++)
   for (j=0; j<m; j++){
      A[i,j] =    ;
             = A[i,j];
      B[i,j+1] =    ;
             = B[i,j];
      C[i+1,j] =    ;
             = C[i,j+1] ;
   }
```



$A_{2,0}=$ $=A_{2,0}$ $B_{2,1}=$ $=B_{2,0}$ $C_{3,0}=$ $=C_{2,1}$
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i
j

A yields: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ B yields: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ C yields: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

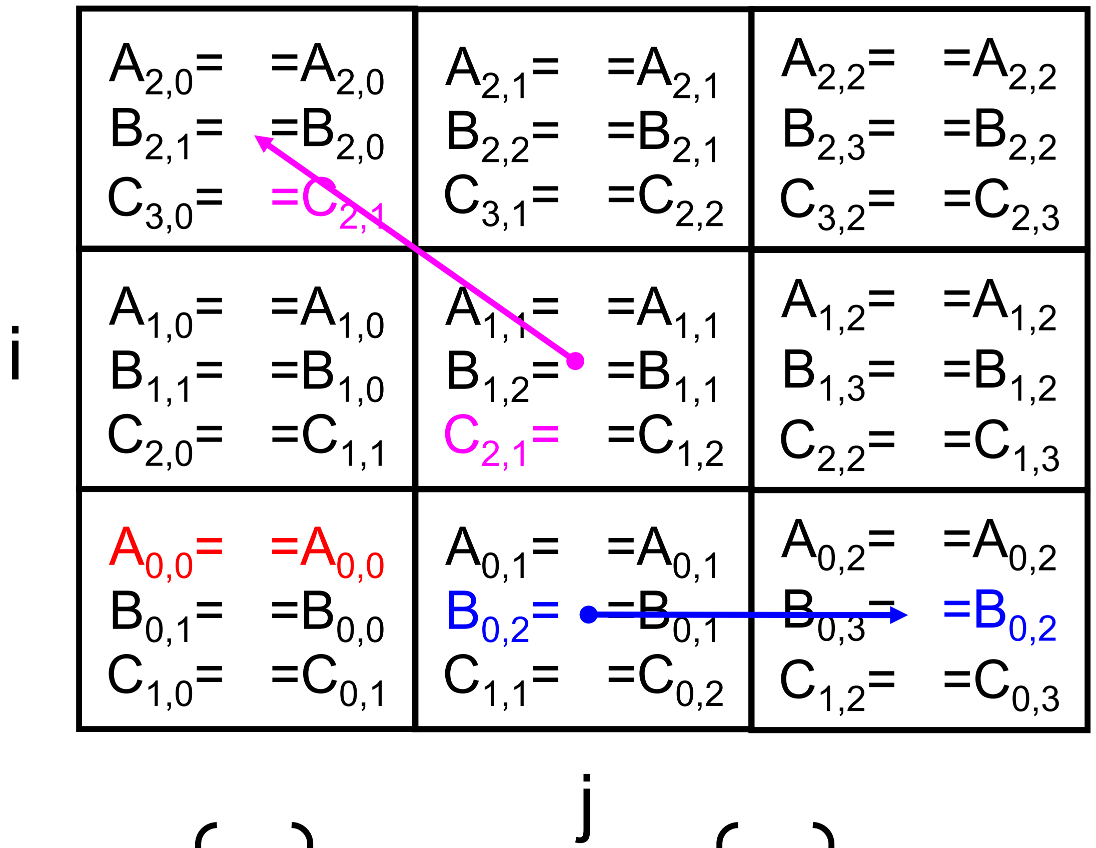
Direction Vectors

- Less precise than distance vectors, but often good enough
- In n-deep loop nest if
  - S1 is source in iteration i
  - S2 is sink in iteration j
- Distance vector: F - Vector of length n, where
  - $f_k = j_k - i_k$
- Direction vector also vector of length n, where
  - $d_k = \begin{cases} \text{“<“ if } f_k > 0, \text{ or } j_k < i_k \\ \text{“=“ if } f_k = 0, \text{ or } j_k = i_k \\ \text{“>“ if } f_k < 0, \text{ or } j_k > i_k \end{cases}$

Sometimes write ‘+’ for < and ‘-’ for >

Example of Direction Vectors

```
for (i=0; i<n; i++)
   for (j=0; j<m; j++){
      A[i,j] =    ;
            = A[i,j];
      B[i,j+1] =    ;
            = B[i,j];
      C[i+1,j] =    ;
            = C[i,j+1] ;
   }
```



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$A_{0,2}=$ $=A_{0,2}$ $B_{0,3}=$ $=B_{0,2}$ $C_{1,2}=$ $=C_{0,3}$
i
j

A yields: $\begin{bmatrix} = \\ = \end{bmatrix}$ B yields: $\begin{bmatrix} = \\ < \end{bmatrix}$ C yields: $\begin{bmatrix} < \\ > \end{bmatrix}$

Another Example

Example:

```
    DO I = 1, N
       DO J = 1, M
          DO K = 1, L
S1            A(I+1, J, K-1) = A(I, J, K) + 10
          ENDDO
       ENDDO
    ENDDO
```

- S_1 has a true dependence on itself.
- Distance Vector: (1, 0, -1)
- Direction Vector: (<, =, >)

Note on vectors

- A dependence cannot exist if it has a direction vector whose leftmost non "=" component is not "<" as this would imply that the sink of the dependence occurs before the source.
- Likewise, the first non-zero distance in a distance vector must be postive.

The Key

- Any reordering transformation that preserves every dependence in a program preserves the meaning of the program

- A reordering transformation may change order of execution but does not add or remove statements.

Finding Data Dependences

The General Problem

```
DO $i_1$ = $L_1$, $U_1$
  DO $i_2$ = $L_2$, $U_2$
                ...
        DO $i_n$ = $L_n$, $U_n$
 $S_1$              A($f_1$($i_1$,...,$i_n$),...,$f_m$($i_1$,...,$i_n$)) = ...
 $S_2$              ... = A($g_1$($i_1$,...,$i_n$),...,$g_m$($i_1$,...,$i_n$))
        ENDDO
        ...
  ENDDO
ENDDO
```

A dependence exists from S1 to S2 if:

– There exist α and β such that

- $\alpha < \beta$ (control flow requirement)
- $f_i(\alpha) = g_i(\beta)$ for all i, $1 \le i \le m$ (common access requirement)

Basics: Conservative Testing

- Consider only linear subscript expressions
- Finding integer solutions to system of linear Diophantine Equations is NP-Complete
- Most common approximation is Conservative Testing, i.e., See if you can assert

  "No dependence exists between two subscripted references of the same array"
- Never incorrect, may be less than optimal

Basics: Indices and Subscripts

Index: Index variable for some loop surrounding a pair of references

Subscript: A PAIR of subscript positions in a pair of array references

For Example:

```
A(I,j) = A(I,k) + C
```

`<I,I>` is the first subscript

`<j,k>` is the second subscript

Basics: Complexity

A subscript is said to be

- ZIV if it contains no index
  zero index variable
- SIV if it contains only one index
  single index variable
- MIV if it contains more than one index
  multiple index variable

For Example:

```
      A(5,I+1,j) = A(1,I,k) + C
            First subscript is ZIV
            Second subscript is SIV
            Third subscript is MIV
```

Basics: Separability

- A subscript is separable if its indices do not occur in other subscripts
- If two different subscripts contain the same index they are coupled

For Example:

```
A(I+1,j)  =  A(k,j)  +  C
```

Both subscripts are separable

```
A(I,j,j)  =  A(I,j,k)  +  C
```

Second and third subscripts are coupled

Basics:Coupled Subscript Groups

- Why are they important?

  Coupling can cause imprecision in dependence testing

```
    DO I = 1, 100
S1     A(I+1,I) = B(I) + C
S2     D(I) = A(I,I) * E
    ENDDO
```

Dependence Testing: Overview

- Partition subscripts of a pair of array references into separable and coupled groups
- Classify each subscript as ZIV, SIV or MIV
  - Reason for classification is to reduce complexity of the tests.
- For each separable subscript apply single subscript test. Continue until prove independence.
- Deal with coupled groups
- If independent, done
- Otherwise, merge all direction vectors computed in the previous steps into a single set of direction vectors

Step 1: Subscript Partitioning

- Partitions the subscripts into separable and minimal coupled groups
- Notations

// S is a set of m subscript pairs S_1, S_2, ...S_m each enclosed in
n loops with indexes I_1, I_2, ... I_n, which is to be
partitioned into separable or minimal coupled groups.

// P is an output variable, containing the set of partitions

// n_p is the number of partitions

Subscript Partitioning Algorithm

```
procedure $partition(S,P, n_p)$
    $n_p = m$;
    for $i$ := 1 to $m$ do $P_i = \{S_i\}$;
    for $i$ := 1 to $n$ do begin
        $k$ := <none>
        for each remaining partition $P_j$ do
            if there exists $s \in P_j$ such that $s$ contains $I_i$ then
                if $k$ = < none > then $k = j$;
                else begin $P_k = P_k \cup P_j$; discard $Pj$; $n_p = n_p - 1$; end
    end
end $partition$
```

Step 2: Classify as ZIV/SIV/MIV

- Easy step
- Just count the number of different indices in a subscript

Step 3: Applying Single Subscript Tests

- ZIV Test
- SIV Test
  - Strong SIV Test
  - Weak SIV Test
    - Weak-zero SIV
    - Weak Crossing SIV
- SIV Tests in Complex Iteration Spaces

ZIV Test

```
  DO j = 1, 100
S       A(e1) = A(e2) + B(j)
  ENDDO
```

e1,e2 are constants or loop invariant symbols

If (e1-e2)!=0 No Dependence exists

Strong SIV Test

- Strong SIV subscripts are of the form

$$\langle ai + c_1, ai + c_2 \rangle$$

- For example the following are strong SIV subscripts

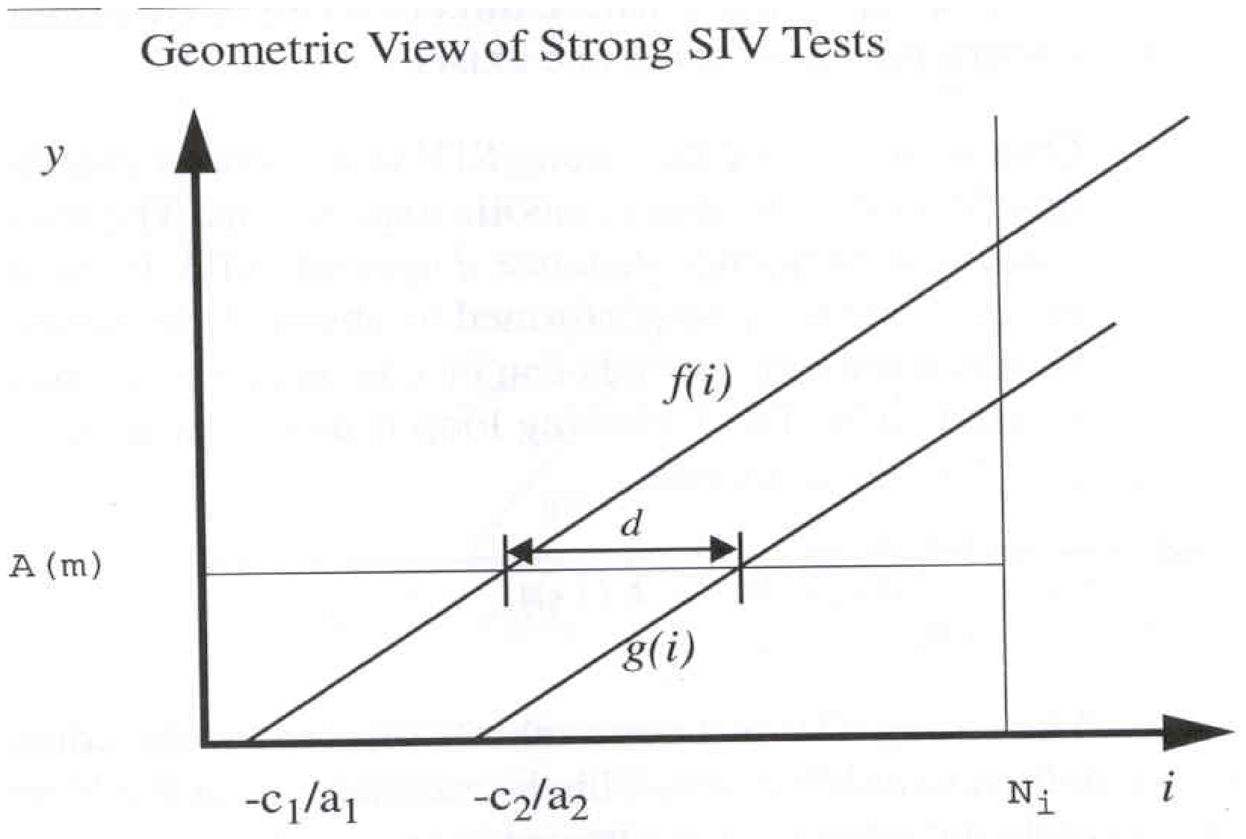
$$\langle i + 1, i \rangle$$

$$\langle 4i + 2, 4i + 4 \rangle$$

Strong SIV Test Example

```
                DO k = 1, 100
                 DO j = 1, 100
       S1             A(j+1,k) = ...
S2                       ... = A(j,k) + 32
                           ENDDO
                         ENDDO
```

Strong SIV Test



$$d = i' - i = \frac{c_1 - c_2}{a}$$

Dependence exists if $|d| \leq U - L$

Weak SIV Tests

- Weak SIV subscripts are of the form

$$\langle a_1 i + c_1, a_2 i + c_2 \rangle$$

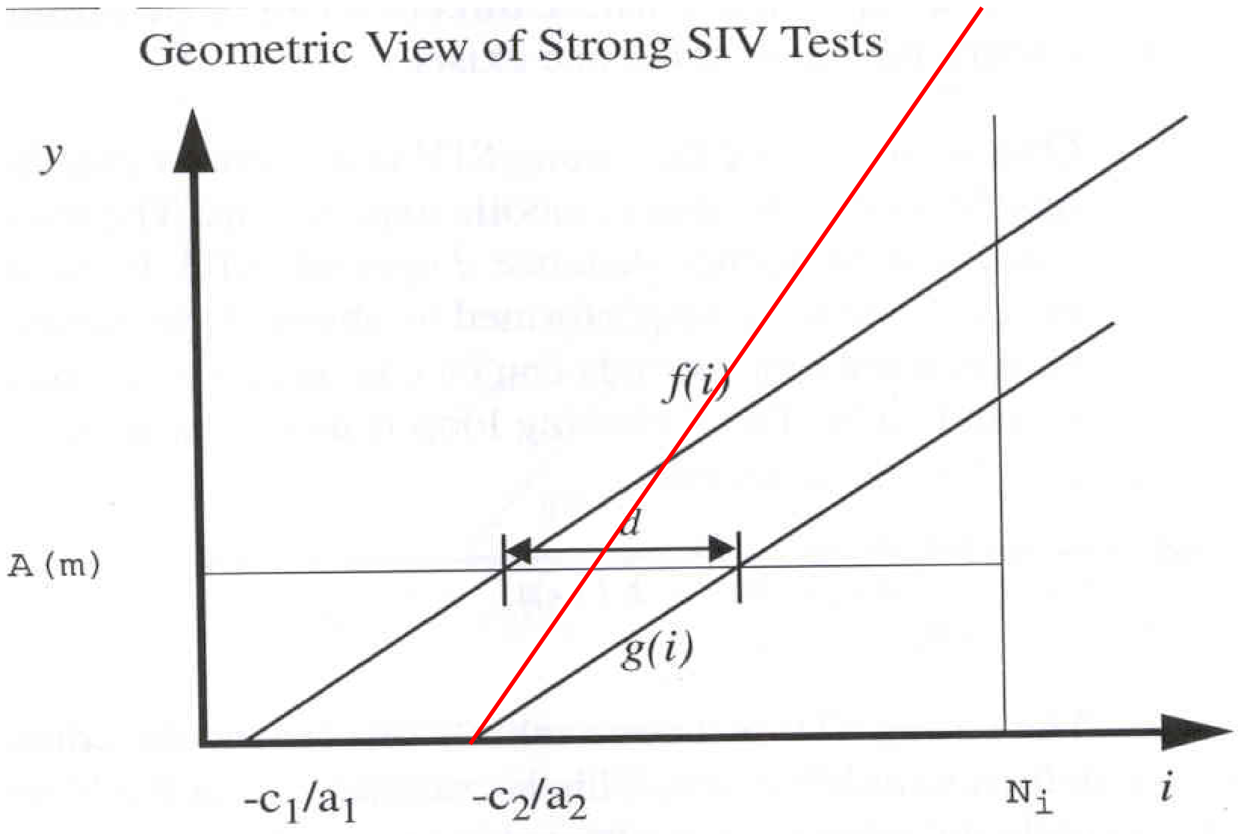
- For example the following are weak SIV subscripts

$$\langle i + 1, 5 \rangle$$

$$\langle 2i + 1, i + 5 \rangle$$

$$\langle 2i + 1, -2i \rangle$$

Geometric view of weak SIV

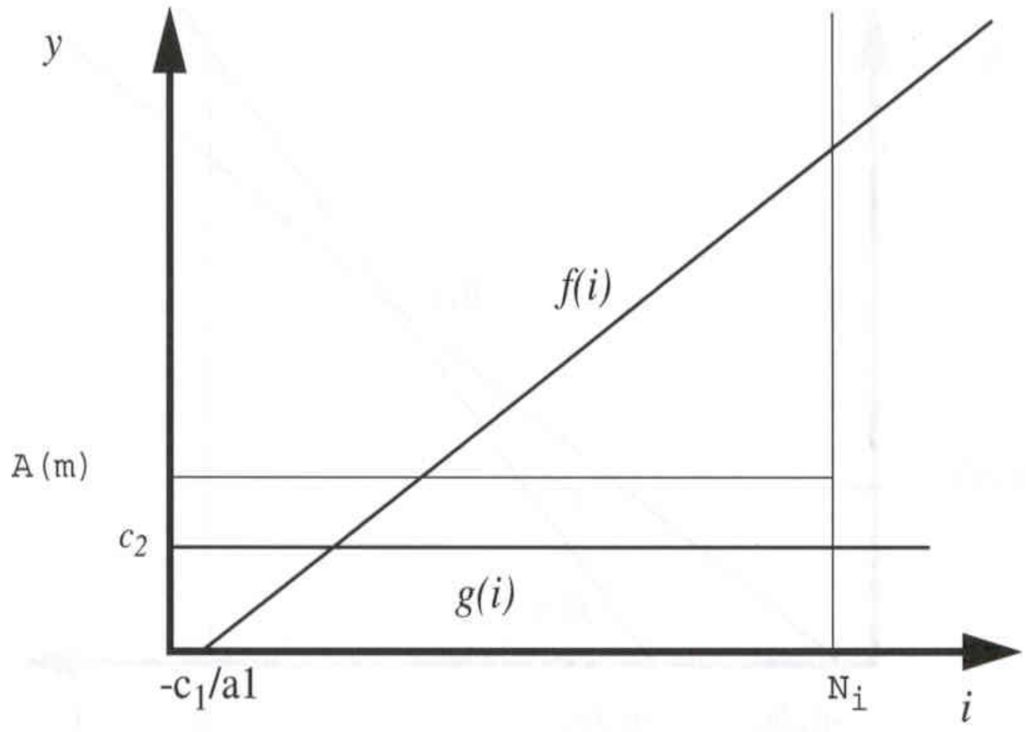


Weak-zero SIV Test

- Special case of Weak SIV where one of the coefficients of the index is zero
- The test consists merely of checking whether the solution is an integer and is within loop bounds $i = \frac{c_2 - c_1}{a_1}$ and, $L \leq i \leq U$

Weak-zero SIV Test

Geometric View of Weak-zero SIV Subscripts



Weak-zero SIV & Loop Peeling

```
      DO i = 1, N
```
S_1
```
         Y(i, N) = Y(1, N) + Y(N, N)
      ENDDO
```

Can be loop peeled to...

```
      Y(1, N) = Y(1, N) + Y(N, N)
      DO i = 2, N-1
S1       Y(i, N) = Y(1, N) + Y(N, N)
      ENDDO
      Y(N, N) = Y(1, N) + Y(N, N)
```

Weak-crossing SIV Test

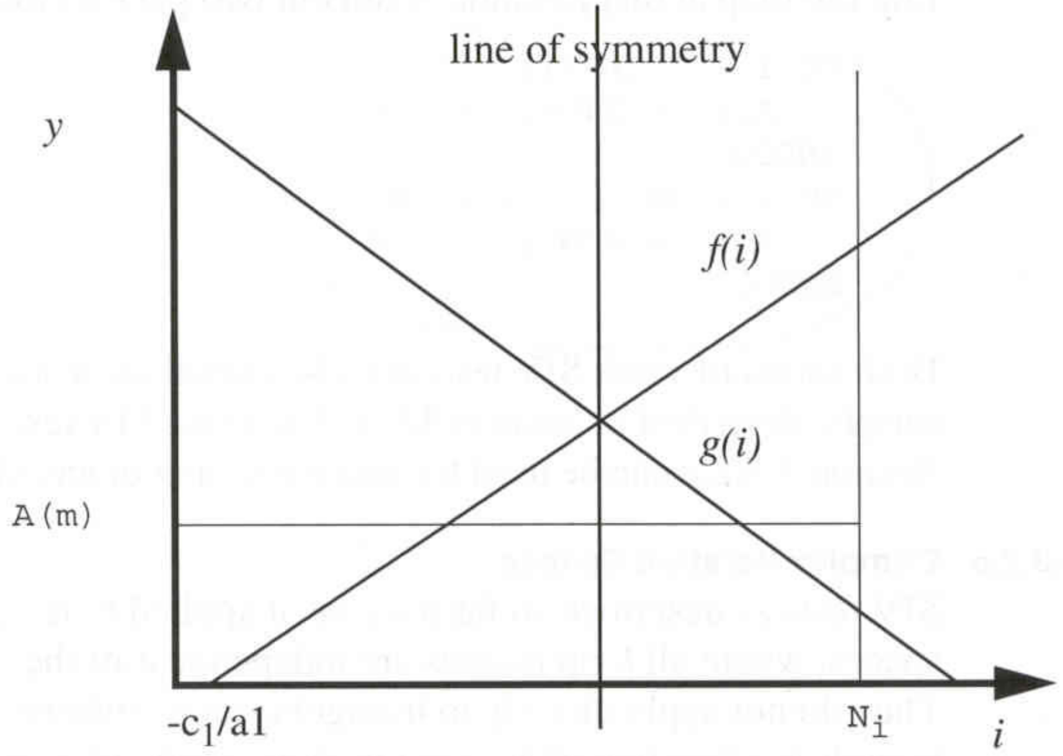
- Special case of Weak SIV where the coefficients of the index are equal in magnitude but opposite in sign
- The test consists merely of checking whether the solution index $i = \frac{c_2 - c_1}{2a_1}$

  is 1. within loop bounds and is

  2. either an integer or has a non-integer part equal to 1/2

Weak-crossing SIV Test

Geometric View of Weak-crossing SIV Subscripts



Weak-crossing SIV & Loop Splitting

```
        DO i = 1, N
S1      A(i) = A(N-i+1) + C
        ENDDO
```

This loop can be split into…

```
        DO i = 1,(N+1)/2
            A(i) = A(N-i+1) + C
        ENDDO
        DO i = (N+1)/2 + 1, N
            A(i) = A(N-i+1) + C
        ENDDO
```

Breaking Conditions

- Consider the following example

```
    DO I = 1, L
S1        A(I + N) = A(I) + B
    ENDDO
```

- If `L<=N`, then there is no dependence from S_1 to itself
- `L<=N` is called the Breaking Condition

Using Breaking Conditions

- Using breaking conditions then can generate alternative code if it would help

```
  IF (L<=N) THEN
   A(N+1:N+L) = A(1:L) + B
  ELSE
   DO I = 1, L
S1        A(I + N) = A(I) + B
   ENDDO
  ENDIF
```

Index Set Splitting

```
DO I = 1,100
  DO J = 1, I
S1      A(J+20) = A(J) + B
  ENDDO
ENDDO
```

For values of $$I < \frac{|d| - (U_0 - L_0)}{U_1 - L_1} = \frac{20 - (-1)}{1} = 21$$

there is no dependence

Index Set Splitting

- This condition can be used to create a part of the loop that is independent

```
    DO I = 1,20
       DO J = 1, I
S1a          A(J+20) = A(J) + B
       ENDDO
    ENDDO
    DO I = 21,100
       DO J = 1, Ix
S1b          A(J+20) = A(J) + B
       ENDDO
    ENDDO
```

Now the inner loop for the first nest is independent.

How are we doing so far?

- Empirical study froom Goff, Kennedy, & Tseng
  - Look at how often independence and exact dependence information is found in 4 suites of fortran programs
  - Compare ZIV, SIV (strong, weak-0, weak-crossing, exact), MIV, Delta
  - Check usefulness of symbolic analysis
- ZIV used 44% of time and proves 85% of indep
- Strong-SIV used 33% of time and proves 5% (success per application 97%)
- S-SIV, 0-SIV, x-SIV used 41%
- MIV used only 5% of time
- Delta used 8% of time, proves 5% of indep
- Coupled subscripts rare (20% overall, but concentrated)

Merging Results

- After we test all subscripts we have vectors for each partition. Now we need to merge these into a set of direction vectors for the memory reference
- Since we partitioned into separable sets we can do cross-product of vectors from each partition.
- Start with a single vector = (*,*,…,*) of length depth of loop nest.
- Foreach parition, for each index involved in vector create new set from

  old vector-these_indicies x this set

Example Merge

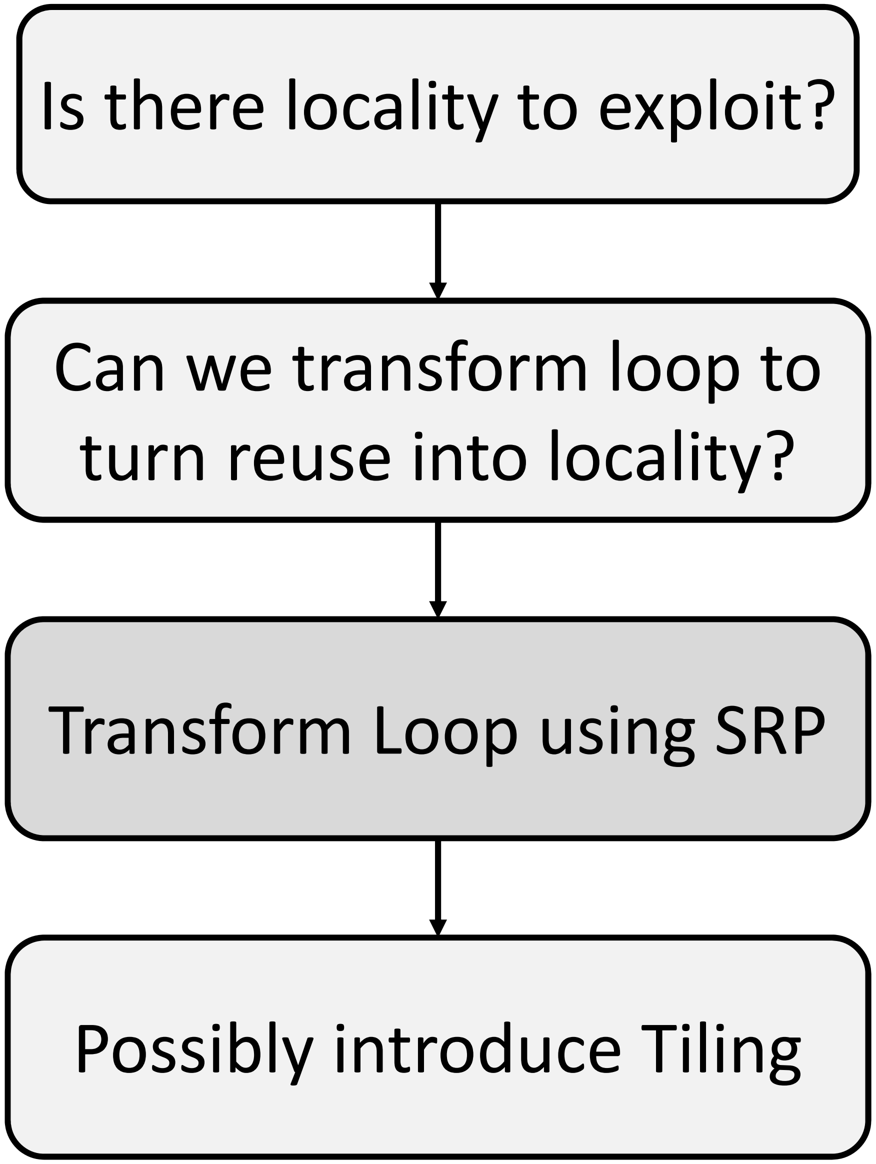
```
  For I
    For J
S_1     A[J-1] = …
S_2     … = A[J]
```

For subscript in A using S_1 as source and S_2 as target: J has DV of -1

Merge -1 into (*,*) -> (*,-1). What does this mean?

- (<,-1): true dep in outer loop
- (=,-1): anti-dep from S_2 to S_1 $\rightarrow$ (=,1)
- (>,-1): anti-dep from S_2 to S_1 in outer loop $\rightarrow$ (<,-1)

Our Goal: Increase locality


Is there locality to exploit?
Can we transform loop to turn reuse into locality?
Transform Loop using SRP
Possibly introduce Tiling

Use Reuse Analysis to determine amount of possible reuse.

Use dependence information to determine pace of possible transformations.

Perform unimodular transformations.

turn n-deep into 2n-deep

Unimodular Transforms

- Interchange

  permute nesting order
- Reversal

  reverse order of iterations
- Skewing

  scale iterations by an outer loop index

Interchange

- Change order of loops
- For some permutation p of 1 … n

```
for $I_1$ := …
    for $I_2$ := …
     …
        for $I_n$ := …
            body
```

$\Rightarrow$

```
for $I_{p(1)}$ := …
    for $I_{p(2)}$ := …
     …
        for $I_{p(n)}$ := …
            body
```

- Legal if permutation on dependence vector is legal

Transform and matrix notation

- If dependences are vectors in iteration space, then transforms can be represented as matrix transforms
- E.g., for a 2-deep loop, interchange is:

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_2 \\ p_1 \end{bmatrix}$$

- Since, T is a linear transform, T**d** is transformed dependence:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$$

Reversal

- Reversal of ith loop reverses its traversal, so it can be represented as:
  Diagonal matrix with ith element = -1.

- For 2 deep loop, reversal of outermost is:

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -p_1 \\ p2 \end{bmatrix}$$

Skewing

- Skew loop I_j by a factor f w.r.t. loop I_i maps

$$(p_1,\ldots,p_i,\ldots,p_j,\ldots) \qquad (p_1,\ldots,p_i,\ldots,p_j+fp_i,\ldots)$$

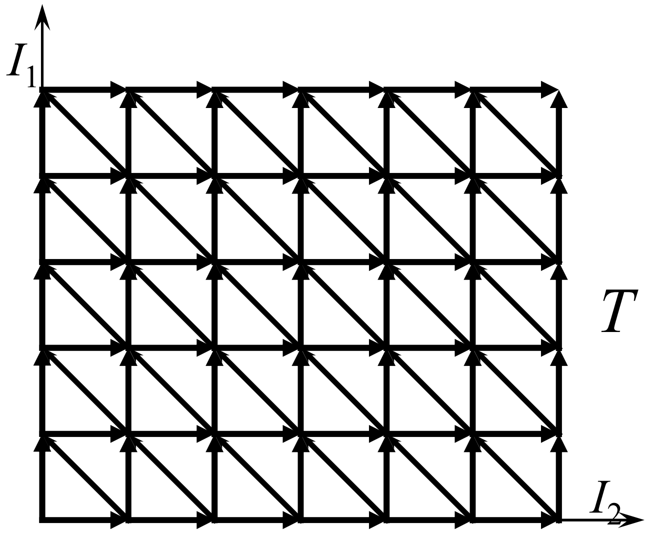
- Example for 2D

$$T=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}p_1\\ p_2+p_1\end{bmatrix}$$

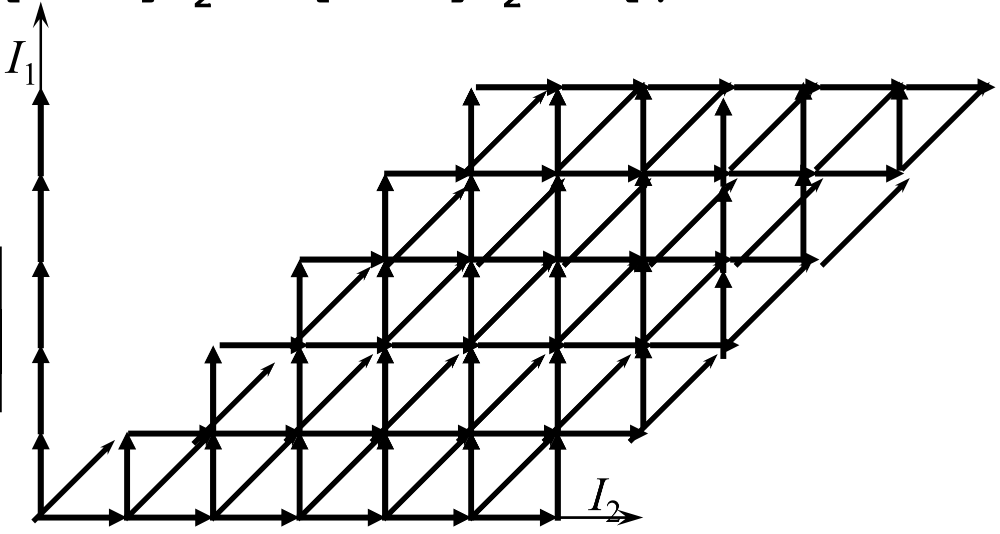
Loop Skewing Example

```
for I_1 := 0 to 5
  for I_2 := 0 to 6
    A[I_2 + 1] := 1/3 * (A[I_2] + A[I_2 + 1] + A[I_2 + 2])
```

D={(0,1),(1,0),(1-1)}



$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



```
for I_1 := 0 to 5
  for I_2 := I_1 to 6+I_1
    A[I_2-I_1+1] := 1/3 * (A[I_2-I_1] + A[I_2-I_1+ 1] + A[I_2-I_1+ 2])
```

D={(0,1),(1,1),(1,0)}

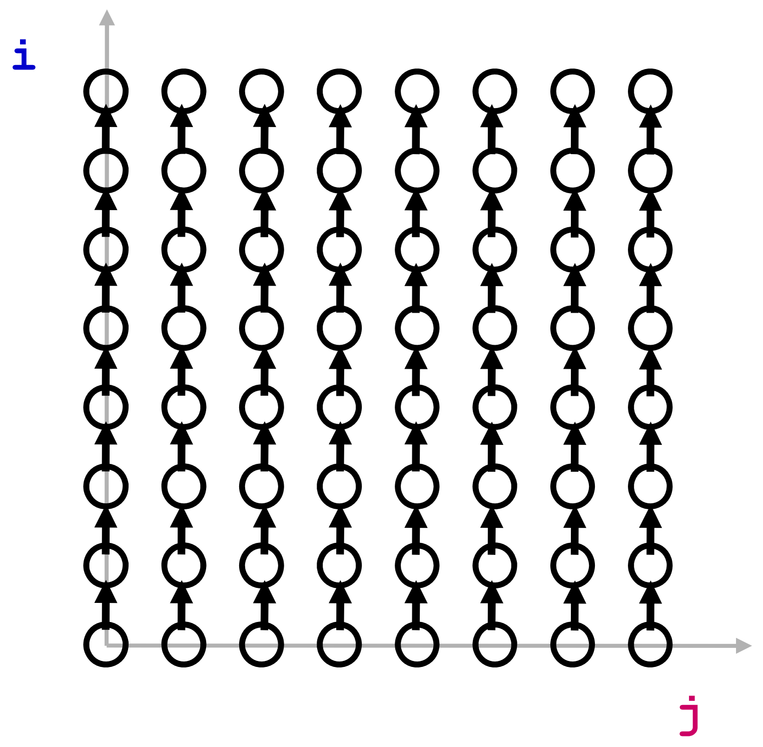
Legal Transformations

- Distance/direction vectors give a partial order among points in the iteration space
- A loop transform changes the order in which 'points' are visited
- The new visit order must respect the dependence partial order!

But…is the transform legal?

- Loop reversal ok?
- Loop interchange ok?

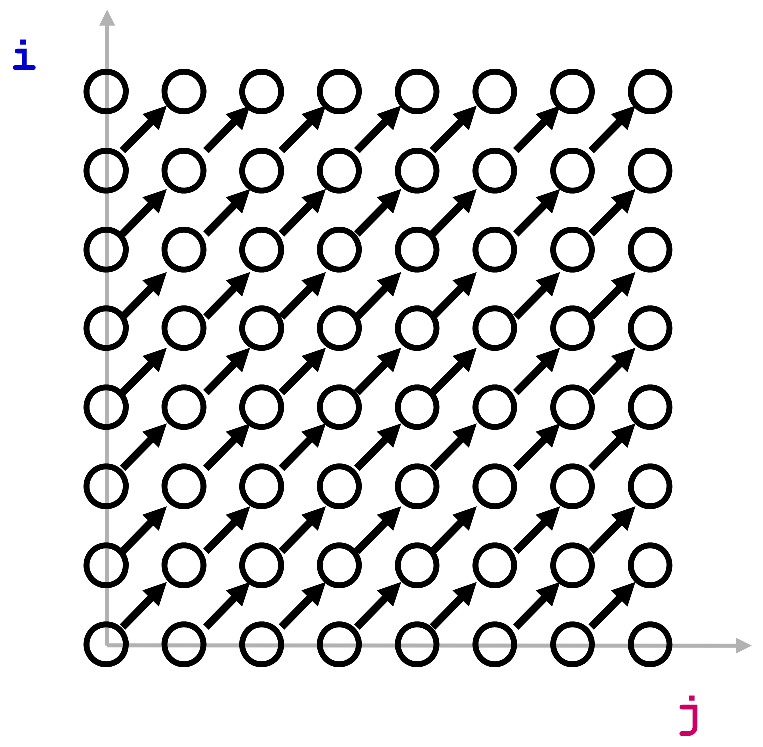
```
for i = 0 to N-1
    for j = 0 to N-1
       A[i+1][j] += A[i][j];
```



But...is the transform legal?

- Loop reversal ok?
- Loop interchange ok?

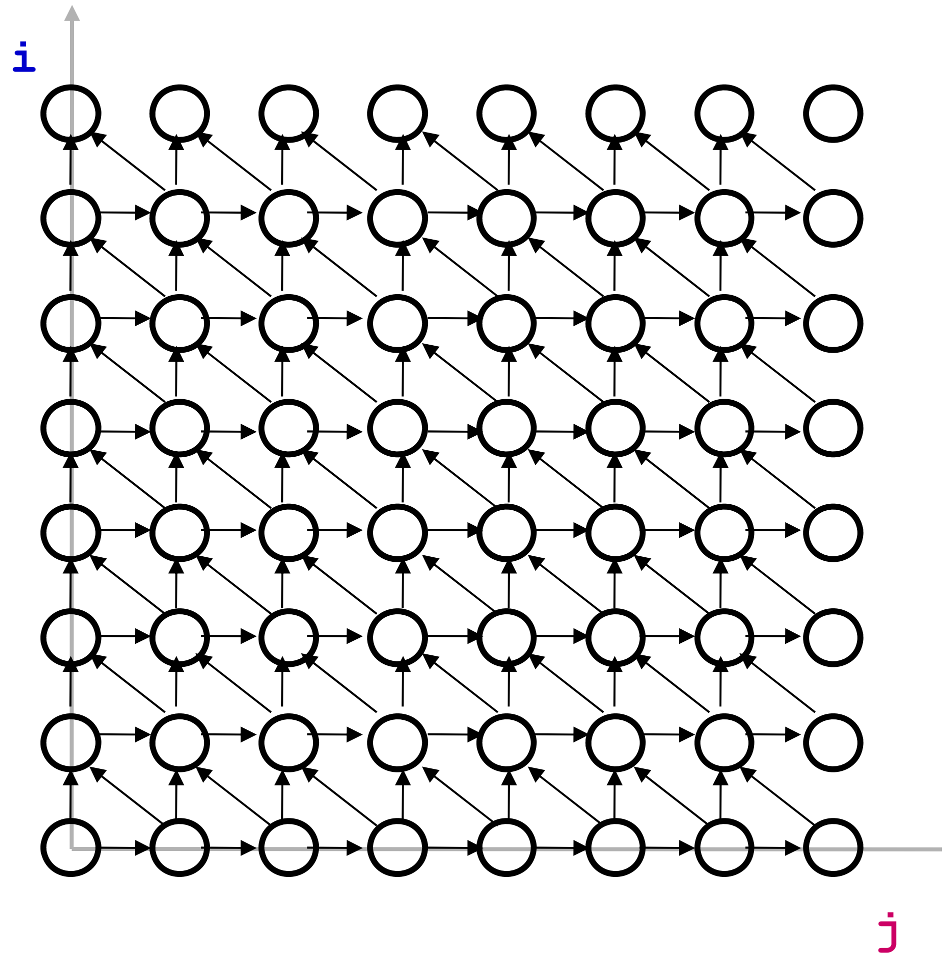
```
for i = 0 to N-1
     for j = 0 to N-1
        A[i+1][j+1] += A[i][j];
```



But...is the transform legal?

- What other visit order is legal here?

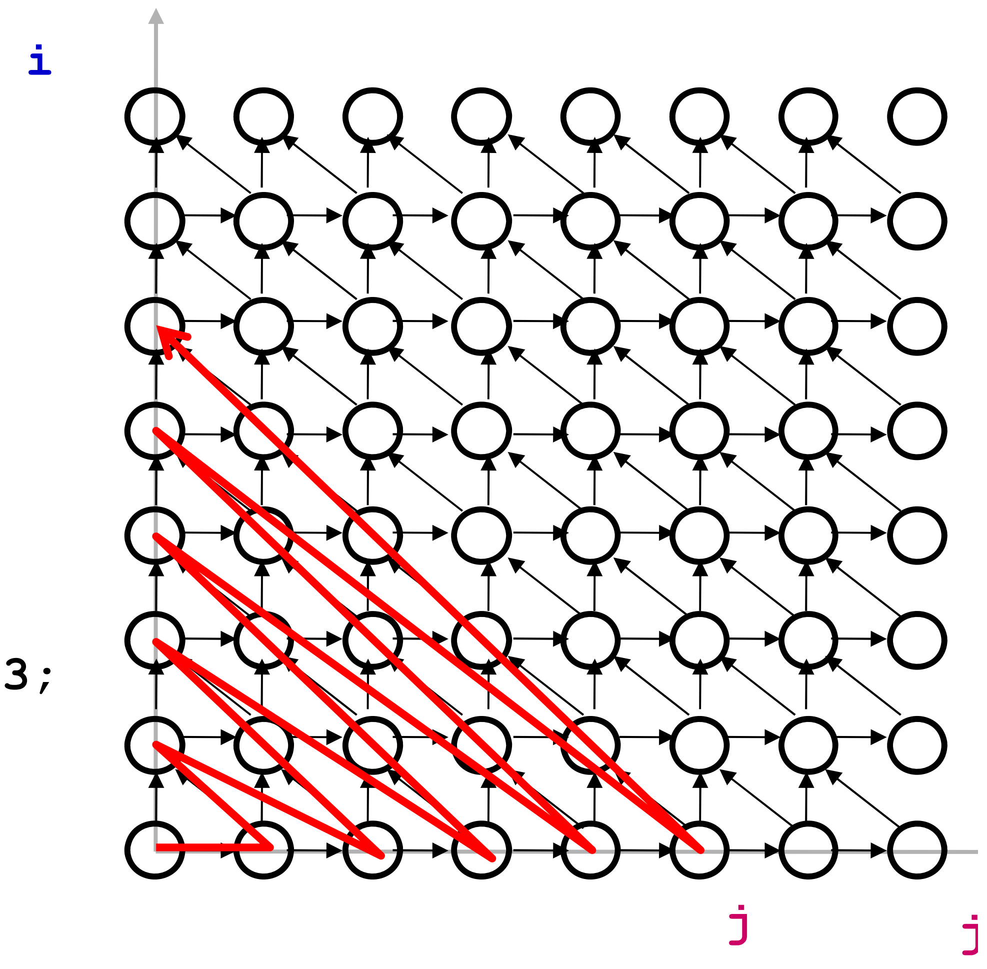
```
for i = 0 to TS
  for j = 0 to N-2
    A[j+1] =
      (A[j] + A[j+1] + A[j+2])/3;
```



But...is the transform legal?

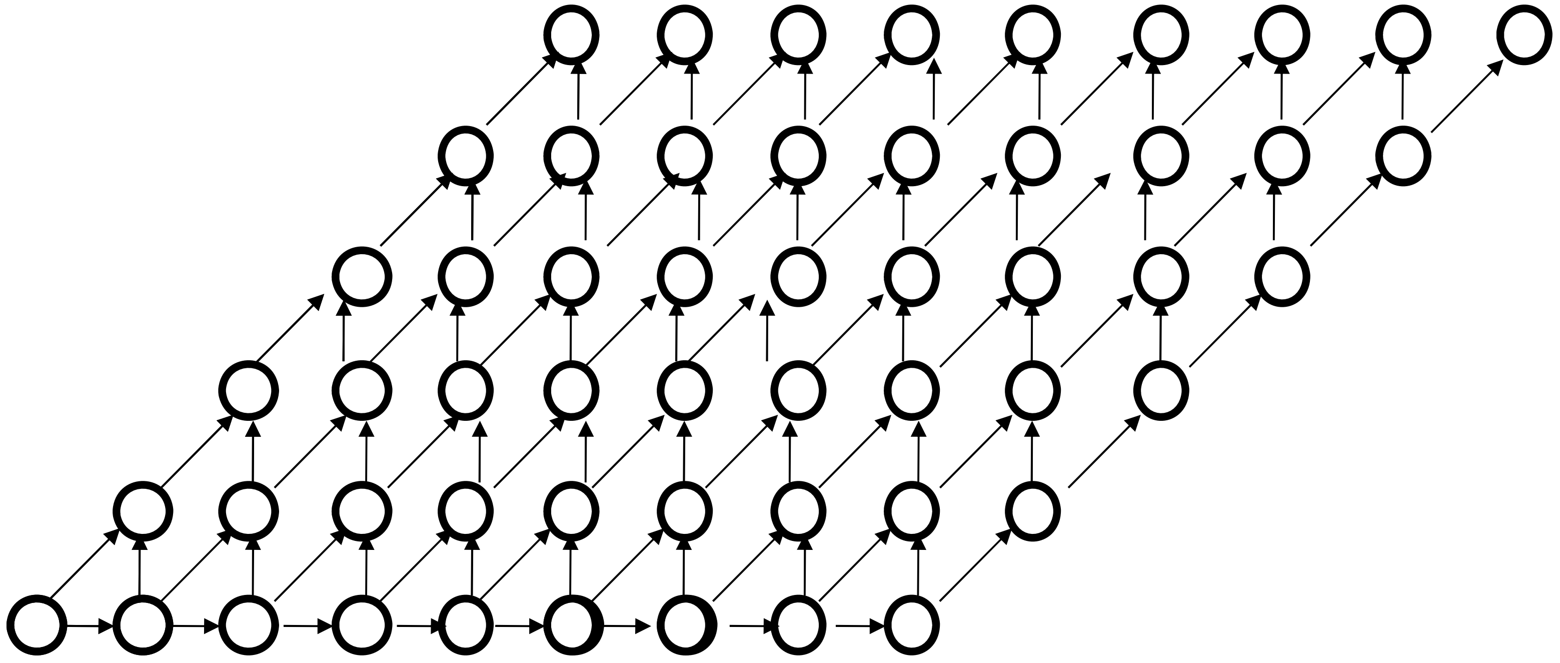
- What other visit order is legal here?

```
for i = 0 to TS
   for j = 0 to N-2
     A[j+1] =
       (A[j] + A[j+1] + A[j+2])/3;
```



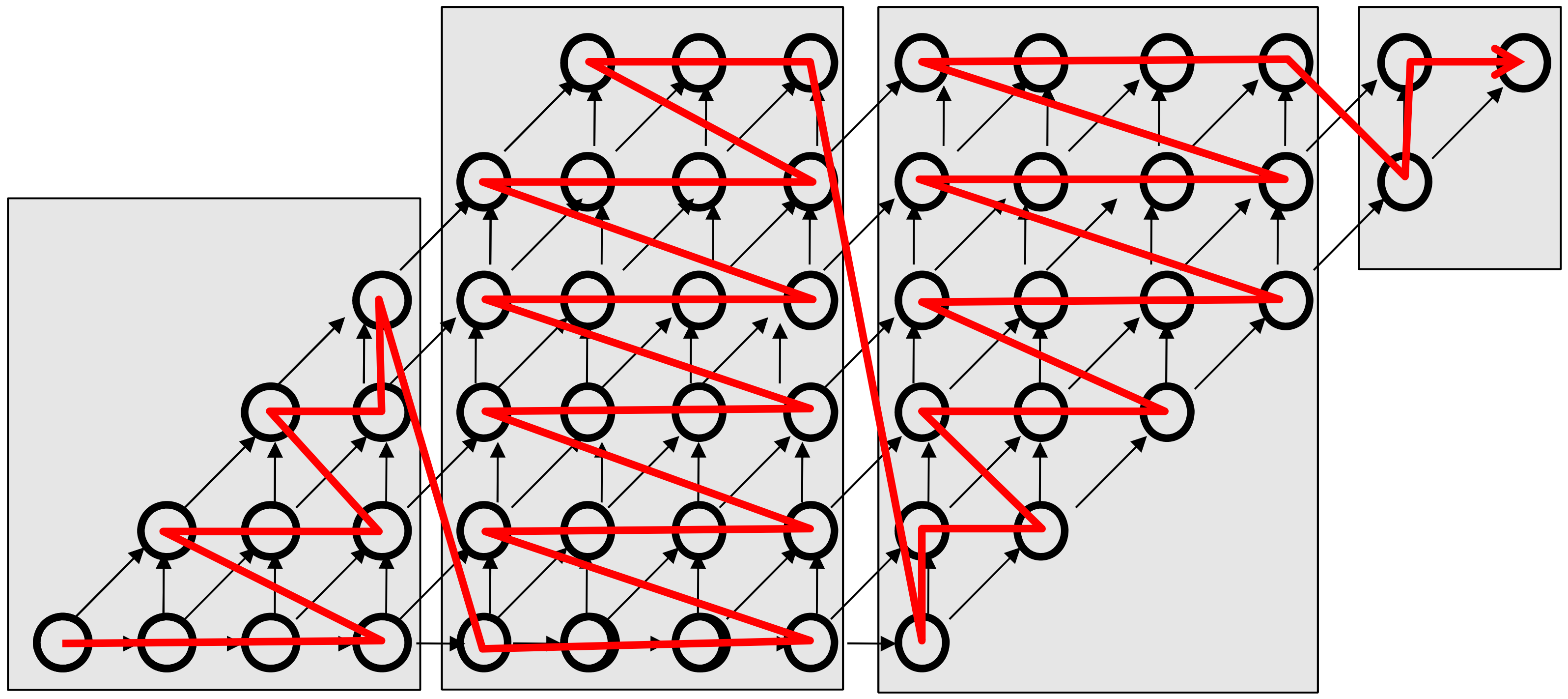
But...is the transform legal?

- Skewing...



But...is the transform legal?

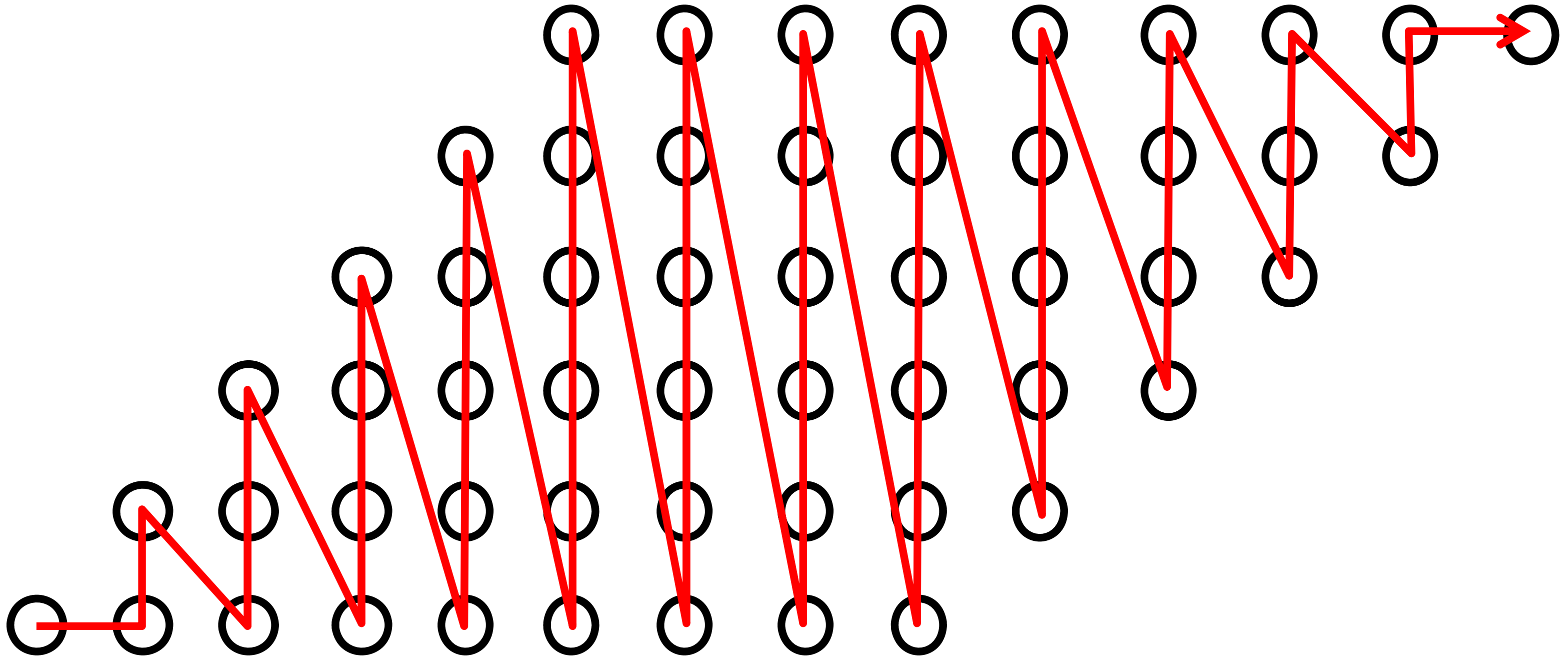
- Skewing...now we can block



We have made the inner loop, Fully Permutable

But...is the transform legal?

- Skewing...now we can loop interchange



Unimodular transformations

- Express loop transformation as a matrix multiplication
- Check if any dependence is violated by multiplying the distance vector by the matrix – if the resulting vector is still lexicographically positive, then the involved iterations are visited in an order that respects the dependence.

Reversal $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Interchange $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Skew $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

"A Data Locality Optimizing Algorithm", M.E.Wolf and M.Lam

SRP

- Extract Dependence Information
- Extract Locality Information
- Search Possible Transformation Space for most Locality

Searching the Space

```
for $I_1$ := 0 to 5
  for $I_2$ := 0 to 6
    A[$I_2$ + 1] = 1/3 * (A[$I_2$]+ A[$I_2$ + 1] + A[$I_2$ + 2])
```

D={(0,1),(1,0),(1,-1)}

Uniformly Generated Set:

{A[I_2], A[I_2+1], A[I_2+2]} $H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Original Loop:

Type	reuse space	reuse factor
Self-Temporal:	Ker(H) = span{(1,0)}	s
Self-Spatial:	Ker(H_s) = span{(1,0),(0,1)}	L
Group-Temporal:	span{(1,0),(0,1)}	3

Possible Transformations

- span{(0,1)} $T=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1/L
- span{(1,0)} illegal
- span{(1,0),(0,1)} $T=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ 1/(sL)

SRP

- Extract Dependence Information
- Extract Locality Information
- Search Possible Transformation Space for most Locality
- Transform Loop using T
  - rewrite index expressions
  - rewrite bounds
- If Neccesary, Tile