MATHEMATICAL FOUNDATIONS OF COMPUTER SCIENCE

TEN POWERFUL IDEAS

and applications

QUIZ 3 Solutions	Bruce Maggs	April 27, 1998

1. [5 points]
Suppose that A and B are events in a probability space, and that $\Pr[A] = 0.5$, $\Pr[B] = 0.2$, and $\Pr[A \cup B] = 0.6$. What is $\Pr[A \cap B]$?

$$\begin{aligned}
\Pr[A \cup B] &= \Pr[A] + \Pr[B] - \Pr[A \cap B] \\
0.6 &= 0.5 + 0.2 - \Pr[A \cap B] \\
\Pr[A \cap B] &= 0.1
\end{aligned}$$

2. [5 points]
Suppose that X is a non-negative integer random variable that takes on the value i with probability $\Pr[X = i]$. What is the definition of $E[X]$, the expectation of X?

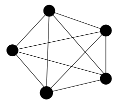
$$E[X] = \sum_{i=0}^{\infty} i \Pr[X = i]$$

3. [5 points]
Suppose that each of the 112 students in 15-299 is assigned uniformly and independently at random to one of four recitation sections. What is the probability that all six students named "Pat" are assigned to the same recitation section?

Here's one way to answer the question. The first Pat can be assigned to any recitation (probability 1), but the second through sixth Pats must be assigned to the same recitation as the first (probability 1/4 for each). So the answer is

$$\left(\frac{1}{4}\right)^5$$

4. [5 points]
Draw all of the 5-node undirected graphs (with no self-loops and no multi-edges) that are not planar.



The only nonplanar 5-node graph is K_5, the complete graph on 5 nodes. (Why are there no others? Removing one edge from K_5 results in a planar graph, which contains every other 5-node graph as a subgraph.)

5. [5 points]
Suppose that an N-node planar graph is drawn in the plane (with no edges crossing) with F faces. How many edges does the graph have?

Euler's formula:

$$E = N + F - 2$$

6. [10 points]
Suppose that A and B are events in a probability space. Using the fact that $A = (A \cap B) \cup (A \cap \overline{B})$, *prove that* $\Pr[A] = \Pr[A|B] \cdot \Pr[B] + \Pr[A|\overline{B}] \cdot \Pr[\overline{B}]$.

$$\begin{aligned}
\Pr[A|B] \cdot \Pr[B] + \Pr[A|\overline{B}] \cdot \Pr[\overline{B}] &= \frac{\Pr[A \cap B]}{\Pr[B]} \Pr[B] + \frac{\Pr[A \cap \overline{B}]}{\Pr[\overline{B}]} \Pr[\overline{B}] \\
&= \Pr[A \cap B] + \Pr[A \cap \overline{B}] \\
&= \Pr[(A \cap B) \cup (A \cap \overline{B})] \text{ since } A \cap B \text{ and } A \cap \overline{B} \text{ are disjoint} \\
&= \Pr[A]
\end{aligned}$$

7. [10 points]
Suppose that a resident of Pittsburgh chosen at random is a Steelers fan with probability 0.90, *that a resident chosen at random is a Penguins fan with probability* 0.75, *and that a resident Steelers fan chosen at random is a Penguins fan with probability* 0.80. *Now suppose that a resident is chosen at random from among those that are not Steelers fans. What is the probability that such a resident is a Penguins fan?*

Let P represent the event that a resident is a Penguins fan, and S represent the event that the resident is a Steelers fan. Then:

$$\begin{aligned}
\Pr[P] &= \Pr[P|S] \Pr[S] + \Pr[P|\overline{S}] Pr[\overline{S}] \\
\Pr[P|\overline{S}] &= \frac{\Pr[P] - \Pr[P|S] * \Pr[S]}{Pr[\overline{S}]} \\
&= \frac{0.75 - 0.80 \cdot 0.90}{1 - 0.9} \\
&= 0.3
\end{aligned}$$

8. [10 points]
Suppose that a fair coin is tossed 100 times. What is the expected number of pairs of consecutive flips in which the coin takes on the same value?

Define a random variable X_i for each pair of consecutive flips, where $X_i = 1$ if flip i and flip $i + 1$ are identical (HH or TT), and $X_i = 0$ if they are different (HT or TH). Thus the expected value of each X_i is $1/2$. The number of identical pairs in the entire sequence is just the sum of the X_i over the entire sequence. By linearity of expectation, the expected value of this quantity is just the sum of the $E[X_i]$. Since there are 99 consecutive pairs, the expected number of identical pairs is

$$49.5$$

9. [10 points]
Prove that the average node degree in a finite tree is less than two.

For any tree with n nodes and e edges, we must have $n = e + 1$. The sum of the degrees of the nodes is $2e$ (since summing the degrees counts both ends of each edge). Thus the average node degree is $2e/n = 2e/(e + 1) < 2$.

10. [15 points]
Prove that if the maximum degree in a finite tree is d, then the tree must have at least d nodes of degree one.

Here are two simple proofs for this fact. The first proof is graphical. If the maximum degree in a tree is d, there must be some node with degree d. Each of this node's d outgoing edges is the start of a path that eventually ends at a leaf, because the tree is finite and acyclic. Furthermore, leaves reached through different neighbors must be different, because the path between two nodes in a tree is unique. Thus the tree must have at least d leaves.

The second proof is algebraic. Suppose the tree has x leaves, one node of degree d, and y other nodes (degree ≥ 2), for a total of $x + y + 1$ nodes. Then the sum of node degrees is at least $x + d + 2y$. From the previous problem, we know that the sum of the degrees is $2e$, that $e = (x + 1 + y) - 1 = x + y$. Combining all these facts gives

$$\begin{aligned} 2e &\geq x + d + 2y \\ 2(x+y) &\geq x + d + 2y \\ x &\geq d \end{aligned}$$

11. [20 points]
Suppose that an n-node random undirected graph is formed as follows. For each pair of distinct vertices u and v, the edge $\{u, v\}$ is present with probability p and not present with probability $1 - p$, independent of whether any other edge is or is not in the graph. What should the probability p be set to so that the expected number of cliques of size k is one?

First we compute the expected number of cliques of size k, then solve for p.

The strategy for computing the expected number of cliques is identical to problem 8 (identical pairs of coin flips): linearity of expectation. Define an indicator random variable for every possible k-clique. Every k-subset of the n nodes is a possible k-clique, so there are $\binom{n}{k}$ such random variables. The random variable for a particular k-clique is 1 if the k-clique is found in the graph and 0 if not, so its expected value is just the probability that a particular k-clique occurs in the graph. This happens if and only if all k nodes are connected by edges; there are $\binom{k}{2}$ such edges, so the probability of a particular k-clique is $p^{\binom{k}{2}}$. Summing the expected values of all the k-cliques gives the expected number of k-cliques:

$$\binom{n}{k} p^{\binom{k}{2}}$$

Setting this quantity to 1 and solving for p gives:

$$p = \left(\frac{1}{\binom{n}{k}}\right)^{\left(\frac{1}{\binom{k}{2}}\right)}$$