10 Powerful Ideas	Bruce Maggs
Quiz 2 Review Exercises	Quiz 2: **Monday, March 16, 1998**

Solutions to these problems will be available on the course Web site by Saturday.

Counting

1. How many 5 card hands (from a 52 card deck) have the same number of Spades and Hearts?

2. How many ways to rearrange the letters "**INSTRUCTOR**"?

3. How many ways to rearrange "**INSTRUCTOR**" with all vowels consecutive?

4. How many ways to rearrange "**INSTRUCTOR**" with no two vowels consecutive?

5. How many integer solutions to the following equation?

$$X_1 + X_2 + X_3 + X_4 + X_5 = 100, \quad 0 \leq X_i \leq 30$$

6. Give a closed-form formula for the sum of the 10th powers of the numbers 1 through n. (If $n = 3$, that would be $1^{10} + 2^{10} + 3^{10} = 60074$.) *Hint: see the end of Lecture 12.*

Infinity

7. Show that $N \times N$ (the set of ordered pairs of natural numbers) is countable.

8. Show that the set of all C programs is countable, or prove that it is not.

9. Recall that the powerset $\mathcal{P}(S)$ of a set S is the set of all subsets of S.

   (a) If $|S| = n$, what is $|\mathcal{P}(S)|$?

   (b) Show that $\mathcal{P}(N)$ is uncountable.

10. Imagine a language with a countably infinite alphabet—in which, for example, anyone could make up a new letter whenever they wanted. Is the set of all strings in this language countable?

Undecidability

Recall that the halting problem is undecidable, i.e., it is impossible to write a program X that takes as its input a description of any program Y and determines whether Y will halt when given its own description as input. I.e., X cannot decide whether $Y(Y)$ will halt for all values of Y.

11. It might seem a little strange to feed a program a description of itself as input, since, after all, very few programs take descriptions of other programs as inputs. So, let's simplify the problem by doing away with Y's input altogether. Is it possible to design a program X' that takes as its input a description of a program Y that contains no input statements, and determines whether Y halts?

12. Suppose that as a compiler writer, you have a more modest goal. You'd like your compiler X to take as its input a program Y (that takes no inputs) and determine whether or not Y initializes all of its variables before they are used. Using a reduction from the halting problem to this problem, show that this problem is still undecidable.

13. How about this problem – is it possible to write a program X'' that takes as its input a C-program Y and determines whether each variable has been declared before it is used in Y?

NP-Completeness

14. How many different 4-cliques does a 10-clique contain?

15. Suppose you are given a subroutine that solves the k-clique decision problem. Show how to use this subroutine to solve the k-clique search problem in polynomial time, assuming that a call to the subroutine takes only one step.

16. In class we noted that the problem of 3-coloring a graph is NP-complete, and that the problem of determining, when given a graph G and a number k as input, whether G has a k-clique, is also NP-complete. What about the problem of determining if a graph has a 3-clique? Show that there is an algorithm for solving this problem in $O(n^3)$ time, where n is the number of vertices in the graph. Is this a contradiction? If not, why not?

17. Ben Bitdiddle, an aspiring 15-299 student, has just made the discovery of a lifetime: Ben has found a polynomial-time reduction from the problem of finding the shortest path in a graph (where all edges have non-negative weights) to the NP-complete problem of 3-coloring a graph. From 15-211, Ben knows that the problem of finding the shortest path in a graph in which all edges have non-negative weights can easily be solved in polynomial time. Before Ben turns his discovery in for extra credit, he wants to run it by you. What's your advice to Ben?

Finite Automata

18. Draw finite automata that accept the following languages over the alphabet $\{0, 1\}$:
    (a) the set of all strings ending 1011
    (b) the set of all strings which are congruent to 0 modulo 3 (when interpreted as integers written in binary notation).

19. Describe (in English) the languages accepted by the following finite automata.