Mathematical Foundations of Computer Science

Ten Powerful Ideas

and applications

QUIZ 1	Steven Rudich	Feb 19, 1996

NAME : Solution Set

ANDREW ID :

SECTION : A B C

INSTRUCTIONS. For the first 10 problems you do not need to give any reason or justifcation for your answer. For the last two problems you must give complete explanations. WRITE CLEARLY. WRITE YOUR NAME AND ANDREW ID ABOVE AND CIRCLE YOUR SECTION.

Repeat After Me.

This part is to test your ability to study material that has been pre-digested and labeled as likely to appear on exams. If you do not do well on this section, your study habits could use adjustment.

1. [5 points]
Give a closed form solution to $3 + 6 + 9 + \ldots + 3n$.

This is just little Gauss's formula, rather poorly disguised:

$$3(1 + 2 + 3 + \ldots n) = \frac{3n(n+1)}{2}$$

2. [5 points]
Give a closed form solution to $\sum_{i=0}^{k} X^i$.

This is known as a *geometric progression*, and you can solve it in almost exactly the same way we showed little Gauss' formula to be true. Let A be the answer. Then

$$\begin{aligned} A &= 1 + X + X^2 + X^3 + \ldots + X^k \\ Ax &= X + X^2 + X^3 + \ldots + X^k + X^{k+1} \end{aligned}$$

Subtracting the first line from the second gives $A(x-1) = X^{k+1} - X$, and solving for A gives $A = \frac{X^{k+1}-1}{x-1}$.

3. [5 points]
Give a formula for n choose k.

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$. This is the number of ways to select k elements from n.

4. [5 points]
How many subsets can be formed from the elements of an n element set?

2^n. For a given subset, each element can either be in the subset or not. So the question is really asking how many different bitstrings of length n are there.

5. [5 points]
What is the formula for the power series expansion of $(x+y)^n$?

$(x+y)^n = \sum_{i=0}^{n} \binom{n}{k} x^k y^{n-k}$

Reading Solutions.

This section tests your ability to read the homework solutions that we hand out.

6. [10 points]
Show a set of bisexual dating preferences where there is no possible stable pairing among the people.

Straight from solutions to homework 2:

person	preference list
A	CBD
B	ACD
C	BAD
D	*doesn't matter*

7. [10 points]
Let $(G, \heartsuit)$ be a finite group. Let a be an element of G. Define $a^0 = e$, where e is the identity element of the group. For natural numbers $n > 0$, define $a^n = a \heartsuit a^{n-1}$. Define $H = \{ a^i \mid i \in \mathbf{N} \}$. We know that H is a subgroup of G. TRUE or FALSE: H is *always* a commutative group.

True. Take any two elements $x, y \in H$. By definition of H, they may be written as $x = a^q$, and $y = a^r$. So $x \heartsuit y = a^q \heartsuit a^r = a^{q+r}$. And $y \heartsuit x = a^r \heartsuit a^q = a^{r+q}$. Since addition is commutative, we know that $a^{q+r} = a^{r+q}$, and so $x = y$.

Basic Techniques.

This part will test your ability to apply techniques for which I have given explicit recipes. If you do not do well on this section, your study habits could use adjustment. You do not need to reduce factorials and binomial coefficients to a numeric answer.

8. [10 points]
How many integer solutions to the following equations?

$$X_1 + X_2 + X_3 + X_4 + X_5 = 15, \quad 0 \leq X_i$$

$\binom{19}{4}$. This is just like the problem of dividing 15 gold bars among 5 thieves: let X_i be the number of gold pieces given to the i'th thief.

Write the gold bars as asterisks, and divide them up by drawing 4 lines. So this is one division:

$$***\,|\,******\,|\,*\,||\,*****$$

Here $X_1 = 3, X_2 = 6, X_3 = 1, X_4 = 0, X_5 = 5$.

(Of course, $\binom{19}{15}$, which is the number of ways to place the asterisks rather than the dividers, is the same thing.)

9. [10 points]
How many 5 card hands with all cards the same suit can be formed from an ordinary 52 card deck?

$4\binom{13}{5}$. Easier question: how many 5-card hands with all hearts? There are 13 hearts, and so there are $\binom{13}{5}$ different ways to select 5 hearts. There's also $\binom{13}{5}$ different ways to select 5 diamonds, etc.

10. [5 points]
How many ways to rearrange the letters in the word EFFORTLESS?

$\binom{10}{2!2!2!}$. There are 10 places to put the first E, 9 places to put the first F, 8 places to put the second F, and so on. But then we've overcounted, since the 2 F's, the 2 E's and the 2 S's are indistinguishable.

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers.

11. [15 points]
A limo is two car lengths long and a car is a single car length long. Let $T(n)$ be the number of distinct possible ways of parking cars and limos along a curb that is n car lengths long. Order matters, but any two cars and any two limos are considered identical. If the curb is 4 car lengths long, then the distinct possible parking patterns are LL, LCC, CLC, CCL, and CCCC. Thus, $T(4) = 5$. Derive a recurrence relation for T that completely determines it. No need to put it in closed form, but justify your answer.

By inspection, $T(0) = 1$ and $T(1) = 1$. The first parked vehicle can be either a car or a limo. If it's a car, then you can fill up the remaining $n - 1$ spots $T(n - 1)$ ways. If it's a limo, then you can fill up the remaining $n - 2$ spots $T(n - 2)$ ways. Hence (by the addition principle), the number of ways to fill up n spots is $T(n - 1) + T(n - 2)$.

Some of you were concerned about overcounting in this argument. But given a sequence of cards and limos, we can use the sleuth's principle to determine the unique set of decisions which led to this arrangement. Look at the first vehicle in the sequence. If it's a car, then we chose a car last, otherwise we chose a limo last. Remove that vehicle and repeat until no vehicles are left.

12. [15 points]
Joe Hacker has a binary encoding scheme capable of representing an arbitrary 5 card poker hand. It uses 30 bits. Each card is described by a six bit code: 2 bits for the suit and 4 bits for the rank. The hand is a 30 bit string listing the five cards in some order. Cindy Cerebral tells Joe that his scheme is wasteful in terms of number of bits used to encode a hand. "I can do it using only 22 bits", she says. "Furthermore, my scheme is the *best possible***". Briefly describe a scheme that is as good as Cindy's scheme and prove that it really is the best possible! Hint: It is a scheme that you know quite well. This problem is much simpler than it first appears.**

There are $\binom{52}{5} = 2,598,960$ different five-card hands we need to encode. But in 22 bits we can encode $2^{22} = 4,194,304$ values. So a simple encoding scheme is just to fix some enumeration of the $\binom{52}{5}$ hands (like we did in assignment 5) and use the 22 bits to communicate the appropriate index.

How do we know there can be no better scheme? Using even one fewer bit (21 bits) we can communicate at most $2^{21} = 2,097,152$ different values. But there are more five-card hands than this, so there can't (by a counting argument) exist a n onto mapping from the numbers $1 \ldots 2^{21}$ to the five-card hands.