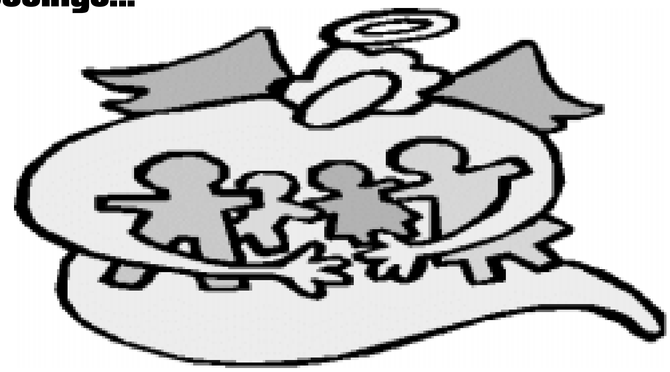
MATHEMATICAL FOUNDATIONS OF COMPUTER SCIENCE

TEN POWERFUL IDEAS

and applications

Count your blessings...



...faster!

15-299 Lecture 11 Feb 17, 1998

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Warm-ups

Q.

How many different ways are there to seat 121 students in a lecture room of 214 chairs?

How many different ways are
there to seat 121 students in a lecture room
of 214 chairs?

There are often multiple ways to count the same set

1. Choose which seats are filled, then order the students in them

$$\binom{214}{121}\cdot 121! = \frac{214!}{121!\cdot 93!}\cdot 121! = \frac{214!}{93!}$$

2. Assign an unfilled seat to each student in a fixed succession, *e.g.* alphabetically.

$$214\cdot 213\cdot 212\cdots(214-121+1) = \frac{214!}{93!}$$

An easier one

Q.

How many different subsets of sleeping students are possible in this class of 121 students?

How many different subsets of sleeping students are possible in this class of 121 students?

There are often multiple interpretations of the same count

$$2^{121}$$

- Subsets of a 121-element set
- Binary digit strings of length 121
- Outcomes of flipping a penny 121 times
- Possible committees drawn from 121 people

Review: The binomial formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{(n-k)}$$

The binomial formula, take two

$$\boxed{(1+x)^n} = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \boxed{\sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k}$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form" or "Generating function"

"Power series" ("Taylor series") expansion

Review: The multinomial formula

$$(x_1 + x_2 + x_3 + \cdots + x_r)^n$$

$$=$$

$$\sum_{\substack{k_1;k_2;k_3;\ldots;k_r \\ k_1+k_2+k_3+\cdots+k_r=n}} \boxed{\binom{n}{k_1;k_2;k_3;\ldots;k_r}} \cdot x_1^{k_1} x_2^{k_2} x_3^{k_3} \cdots x_r^{k_r}$$

$$\boxed{= \frac{n!}{k_1!k_2!k_3!\cdots k_r!}}$$

Multinomial mania

Q.

What Is the coefficient of (M•A•G•G•S) in the expansion of $(S+M+A+G)^5$?

A.

The same as the number of arrangements of "MAGGS", or

$$\frac{5!}{2!} = \binom{5}{1;1;2;1} = 60$$

Representation

Representation

Representation

Explore different possible representations of the same information or idea, and understand the relationship between them.

Playing with the binomial formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let *x=1*. We find that

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Can you explain this *combinatorially?*

Playing with the binomial formula

$$\boxed{2^n} = \boxed{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an *n*-element set

The number of *k*-element subsets of an *n*-element set, summed over all possible *k*.

Indeed, these mean the same thing!

Combinatorial proofs

A combinatorial proof demonstrates that each side of an equation corresponds to the size of the same set.

Contrast this to a conventional algebraic proof, in which symbol manipulation is used to carry one side to the other

More binomial formulations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let *x*= *-1*. We find that

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

…or equivalently, that

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The odds get even

$$\boxed{\sum_{k \text{ even}}^{n} \binom{n}{k}} = \boxed{\sum_{k \text{ odd}}^{n} \binom{n}{k}}$$

The number of length-*n* binary strings with an even number of ones

The number of length-*n* binary strings with an odd number of ones

The algebra has spoken. But it's not yet independently clear why these sides count the same thing. Let's develop a *correspondence* from one to the other.

More odds and evens

Let O_n be the set of` binary strings of length n with an odd number of ones.
Let E_n be the set of` binary strings of length n with an even number of ones.

We have already presented an algebraic proof that $O_n = E_n$

An elegant combinatorial proof can be had by putting O_n and E_n in one-to-one correspondence.

The correspondence principle says that *if two sets can be placed in one-to-one correspondence, then they are the same size*!

An attempt at a correspondence

Let f_n be the function that takes an n-bit bitstring and flips all its bits.

f_n is clearly a one-to-one and onto function for odd n. *E.g.* in f_7 we have

0010011 → 1101100
1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100
101010 → 010101

Uh oh. Complementing maps evens to evens!

A correspondence that works for all *n*

Let f_n be the function that takes an n-bit bitstring and flips only *the first bit*.
For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

***Check*:**

1. $f_n : O_n \to E_n$?
2. f_n is one-to-one? *i.e.* $x \neq y \Rightarrow f_n(x) \neq f_n(y)$
3. f_n is onto? *i.e.* for all $y \in E_n$, there exists an $x \in O_n$ such that $f_n(x)=y$

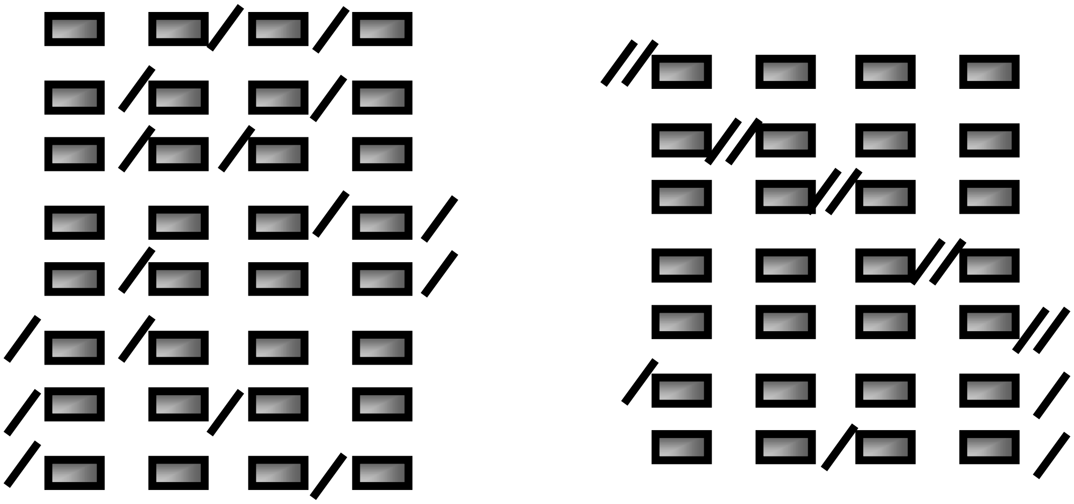
How to count allocation schemes

Example 1: Pirates and gold bars

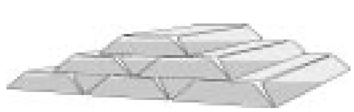
Scenario: You're a pirate who has just discovered n bars of gold (identical and indivisible). Being a generous buc, you decide to split the loot between the k distinct shipmates on board. How many ways are there to do this?

Example: n=4, k=3

Representation: Partition a string of 4 gold bars into 3 substrings by inserting slashes.



… 15 = $\binom{4+3-1}{3-1}$ allocation schemes!



Connecting to a known representation

So the number of allocation schemes *is the same as* the number of strings of bars and slashes with n bars and k-1 slashes…

…which *is the same as* the number of ways to choose k-1 positions to make slashes from a set of n+k-1 positions, or

$$\binom{n+k-1}{k-1}$$

How to count allocation schemes
Example 2:
Solutions to integer equations

Q. How many ways are there to solve:

$$x_1 + x_2 + x_3 = 10$$

$$x_1, x_2, x_3 \geq 0$$

A. It's *the same as* distributing 10 gold bars to 3 pirates!

$$\binom{10+3-1}{3-1} = \binom{12}{2} = 66$$

How to count allocation schemes
Example 3:
Solutions to constrained integer equations

Q. A twist: what if the solutions must be strictly positive?

$$x_1 + x_2 + x_3 = 10$$

$$x_1, x_2, x_3 > 0$$

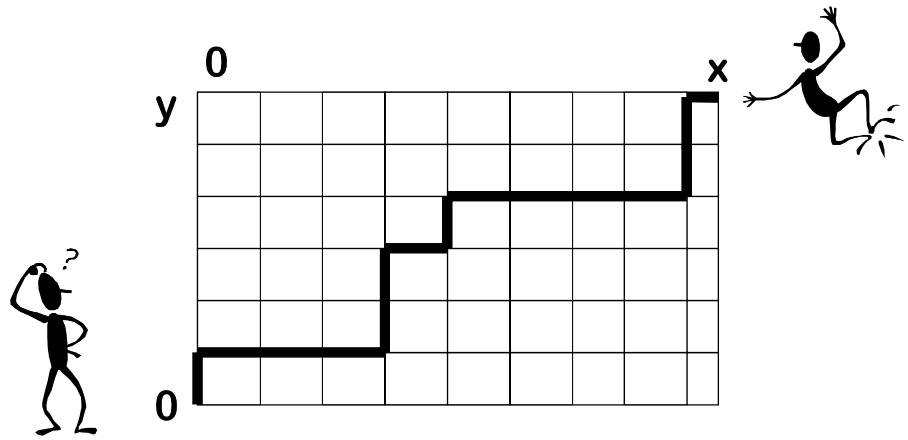
A. First give every "pirate" his required 1 "gold bar".
Then count the ways to distribute the remaining 10-3=7 "gold bars":

$$\binom{7+3-1}{3-1} = \binom{9}{2} = 36$$

How to count pathways

Meandering in a nameless modern metropolis

Scenario: You're in a city where all the streets, numbered 0 through x, run north-south, and all the avenues, numbered 0 through y, run east-west. How many [sensible] ways are there to walk from the corner of 0th St. and 0th avenue to the opposite corner of the city?

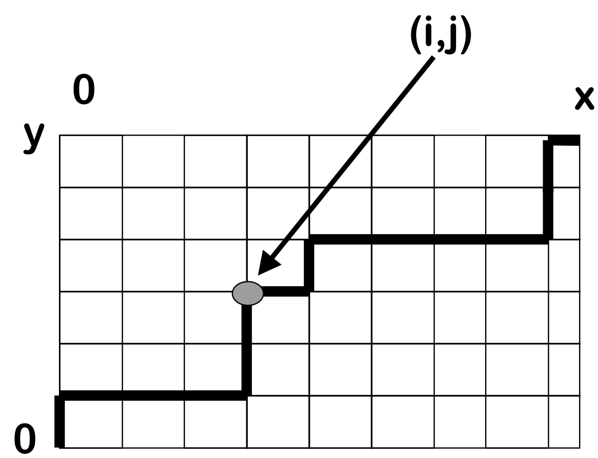


Meandering in a nameless modern metropolis

- All paths require exactly x+y steps:
- x steps east, y steps north
- Counting paths is the same as counting which of the x+y steps are northward steps:

$$\binom{x+y}{y}$$

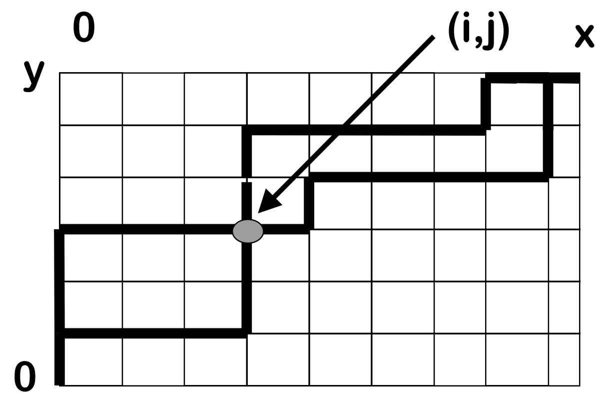
Now, what if we add the constraint that the path must go through a certain intersection, call it (i,j)?



Meandering in a nameless modern metropolis

- Given the constraint, we can decompose each valid path into two subpaths:
- The subpath from the start to (i,j) $\binom{i+j}{i}$
- The subpath from (i,j) to (y,x) $\binom{(y-i)+(x-j)}{(y-i)}$
- These subpaths may be independently chosen. By the product rule, the total path count is

$$\binom{i+j}{i}\cdot\binom{(y-i)+(x-j)}{(y-i)}$$

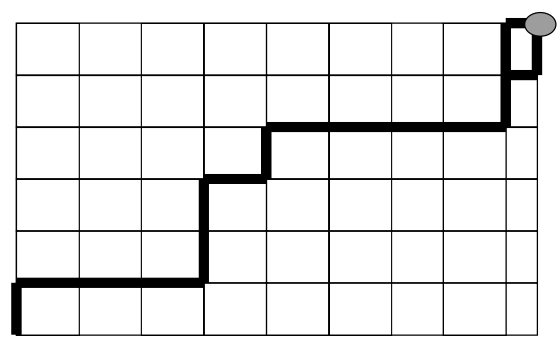


An important identity for binomial coefficients

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

Combinatorial proof? Consider separating all k-element subsets of the set {1,2,…,n} into those that include and those that exclude n

Graphical intuition:
Let n=x+y be the total steps needed in the city walk problem, and let k=y be the number of northward steps. There are two cases for the very last step taken.



Toward Pascal’s Triangle

Associate with each intersection the path count from (0,0), $\binom{n}{k}$

$\binom{5}{5}$					
$\binom{4}{4}$	$\binom{5}{4}$				
$\binom{3}{3}$	$\binom{4}{3}$	$\binom{5}{3}$			
$\binom{2}{2}$	$\binom{3}{2}$	$\binom{4}{2}$	$\binom{5}{2}$		
$\binom{1}{1}$	$\binom{2}{1}$	$\binom{3}{1}$	$\binom{4}{1}$	$\binom{5}{1}$	
$\binom{0}{0}$	$\binom{1}{0}$	$\binom{2}{0}$	$\binom{3}{0}$	$\binom{4}{0}$	$\binom{5}{0}$

Toward Pascal’s Triangle

Simplifying, we observe startling symmetries

1					
1	5				
1	4	10			
1	3	6	10		
1	2	3	4	5	
1	1	1	1	1	1

Pascal's Triangle

Credited to
Blaise Pascal, 1654

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

- Blaise

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\iddots & & & & & & \vdots & & & & & & \ddots
\end{array}
$$

Summing the rows... gives us powers of 2

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$
\begin{array}{rl}
1 & =1 \\
1 + 1 & =2 \\
1 + 2 + 1 & =4 \\
1 + 3 + 3 + 1 & =8 \\
1 + 4 + 6 + 4 + 1 & =16 \\
1 + 5 + 10 + 10 + 5 + 1 & =32 \\
1 + 6 + 15 + 20 + 15 + 6 + 1 & =64 \\
\iddots \quad \vdots \quad \ddots &
\end{array}
$$

Summing the diagonals…
yields Little Gauss's formula and more!

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
⋰          ⋮          ⋱
```

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

- Blaise

Try your hand.