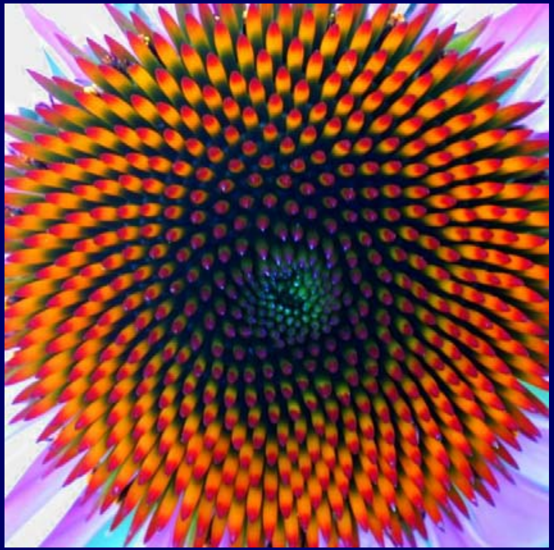
John Lafferty CS 15-251 Fall 2006

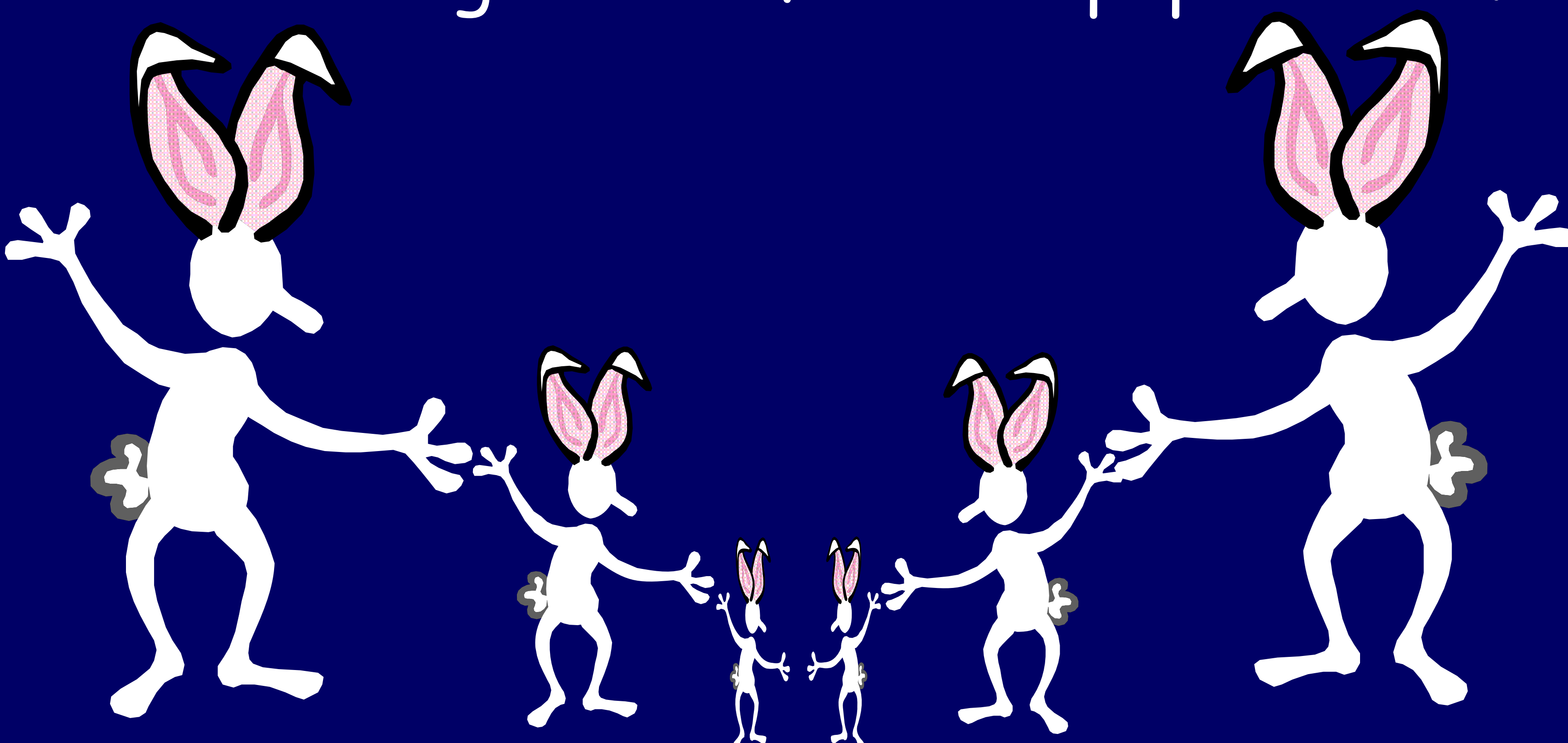
Lecture 13 October 10, 2006 Carnegie Mellon University

The Golden Ratio, Fibonacci Numbers, And Other Recurrences



Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.



Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:

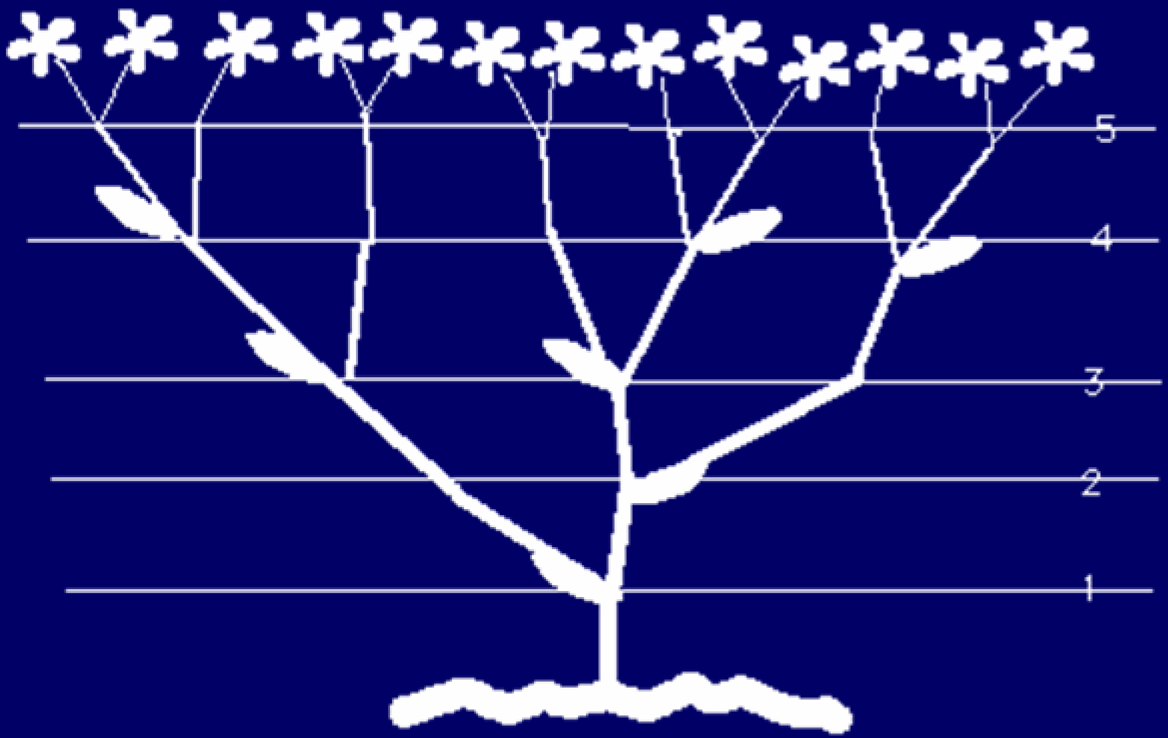
Fib(0) = 0; Fib (1) = 1

Inductive Rule

For n>1, Fib(n) = Fib(n-1) + Fib(n-2)

n	0	1	2	3	4	5	6	7
Fib(n)	0	1	1	2	3	5	8	13

Sneezwort (Achilleaptarmica)



Each time the plant starts a new shoot it takes two months before it is strong enough to support branching.

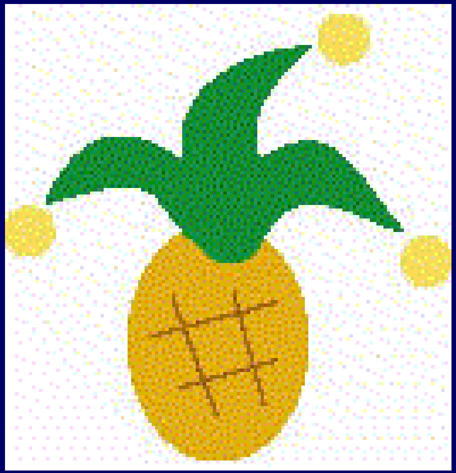
Counting Petals

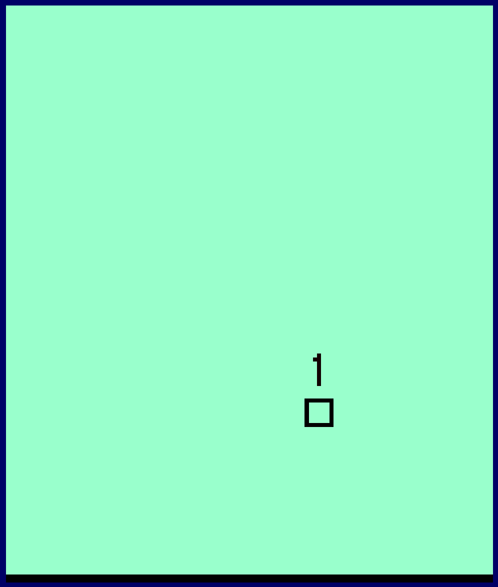
5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)
8 petals: delphiniums
13 petals: ragwort, corn marigold, cineraria, some daisies
21 petals: aster, black-eyed susan, chicory
34 petals: plantain, pyrethrum
55, 89 petals: michaelmas daisies, the asteraceae family.

Pineapple whorls

Church and Turing were both interested in the number of whorls in each ring of the spiral.

The ratio of consecutive ring lengths approaches the Golden Ratio.

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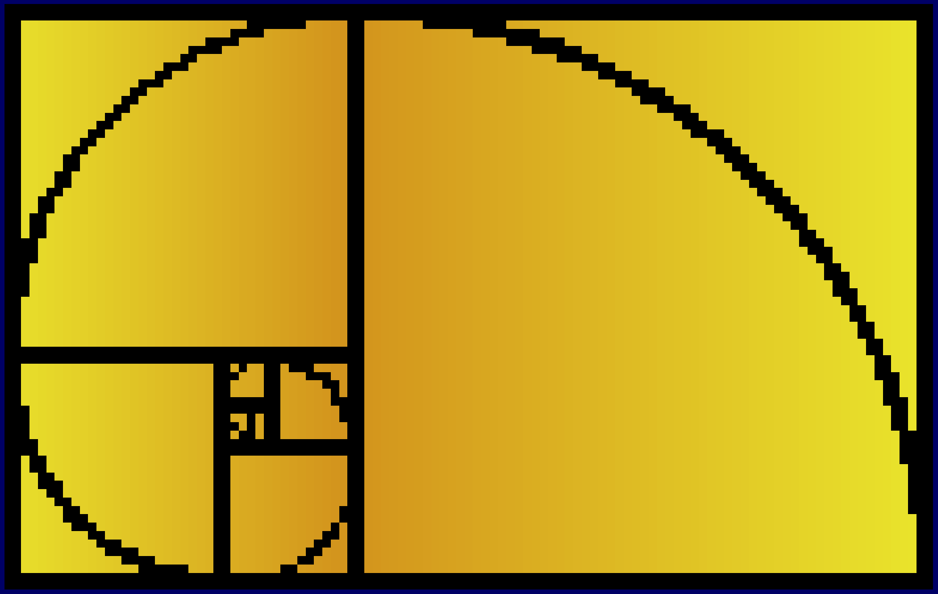
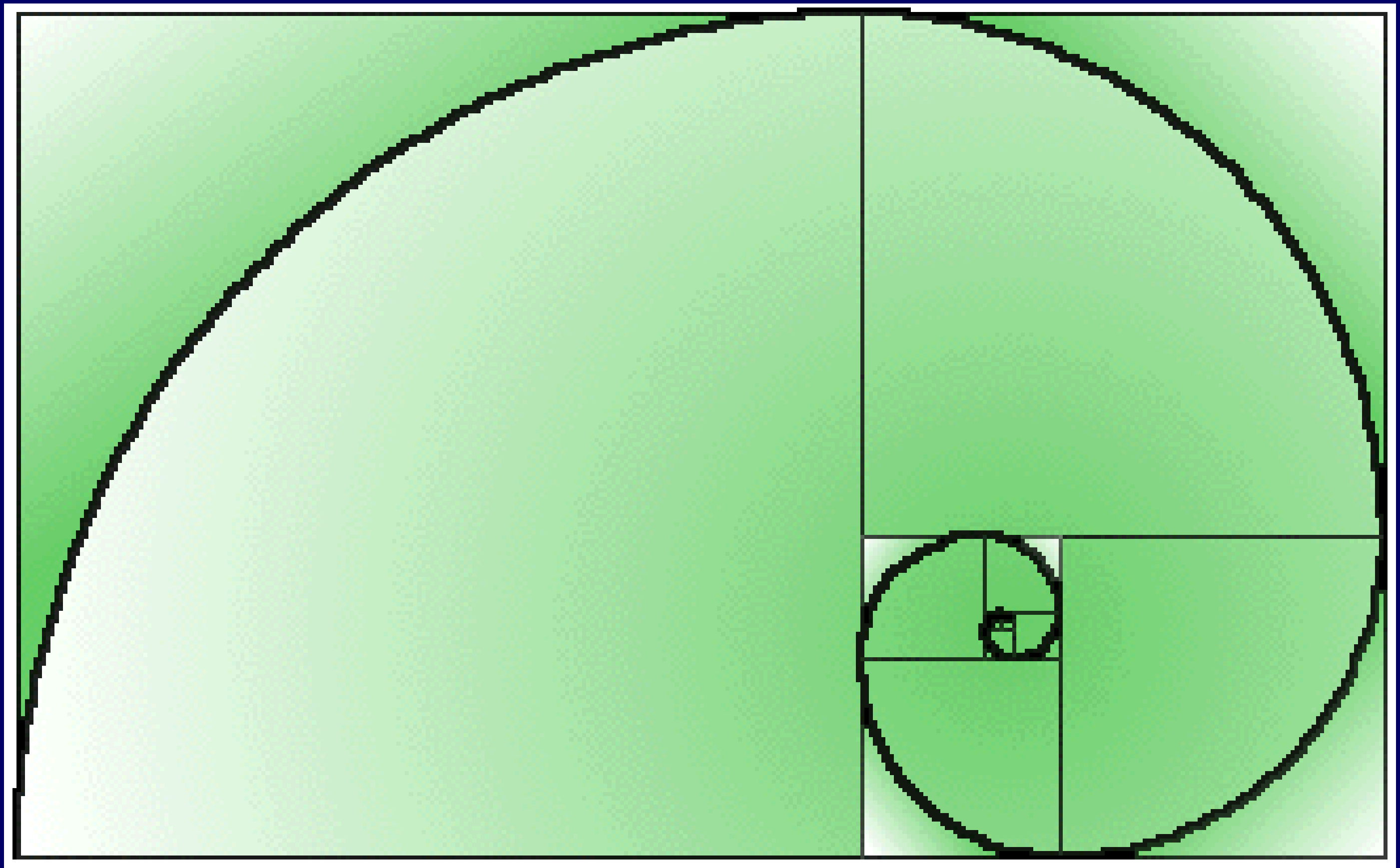


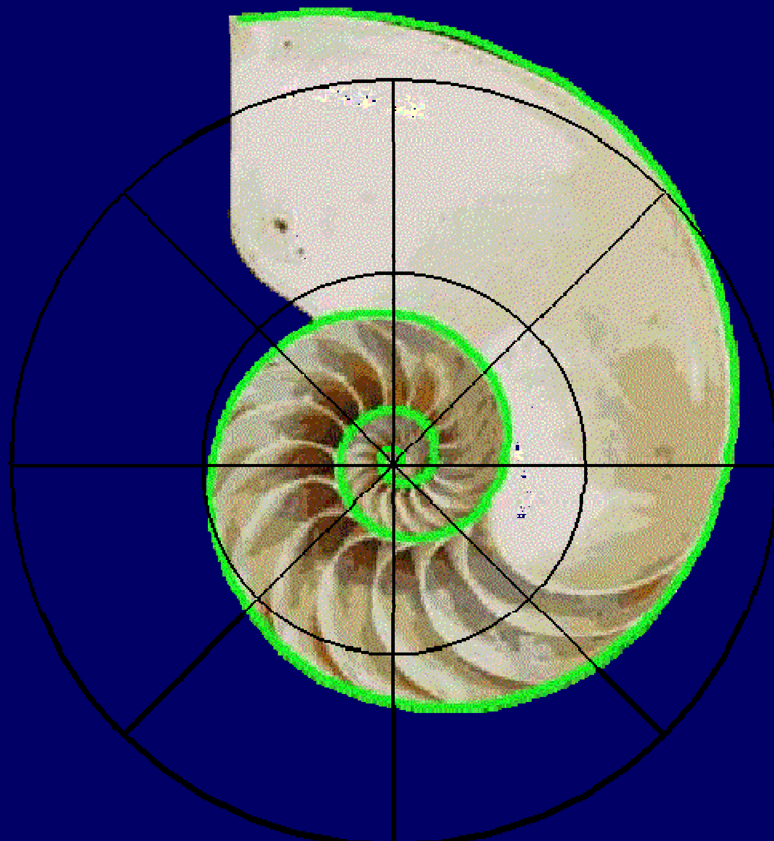
Bernoulli Spiral

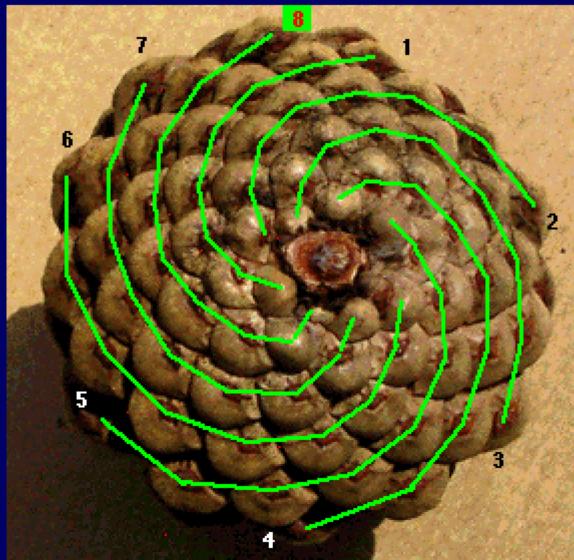
When the growth of the organism is proportional to its size

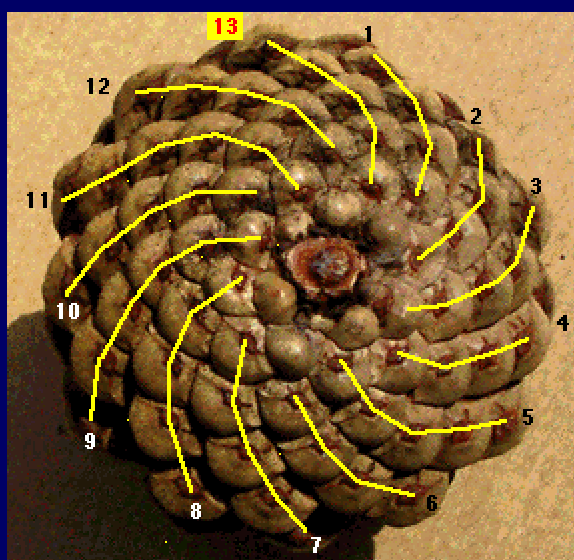


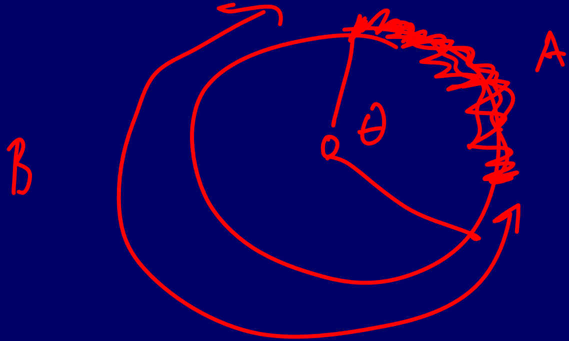
Bernoulli Spiral

When the growth of the organism is proportional to its size




8
7
1
6
2
5
3
4


13
1
12
2
11
3
10
4
9
5
8
6
7



$$\frac{B}{A} = \phi$$

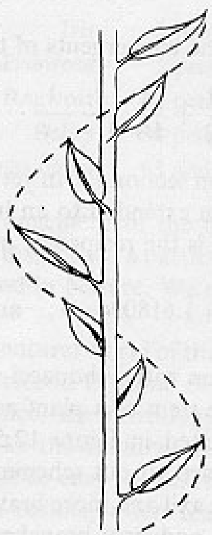
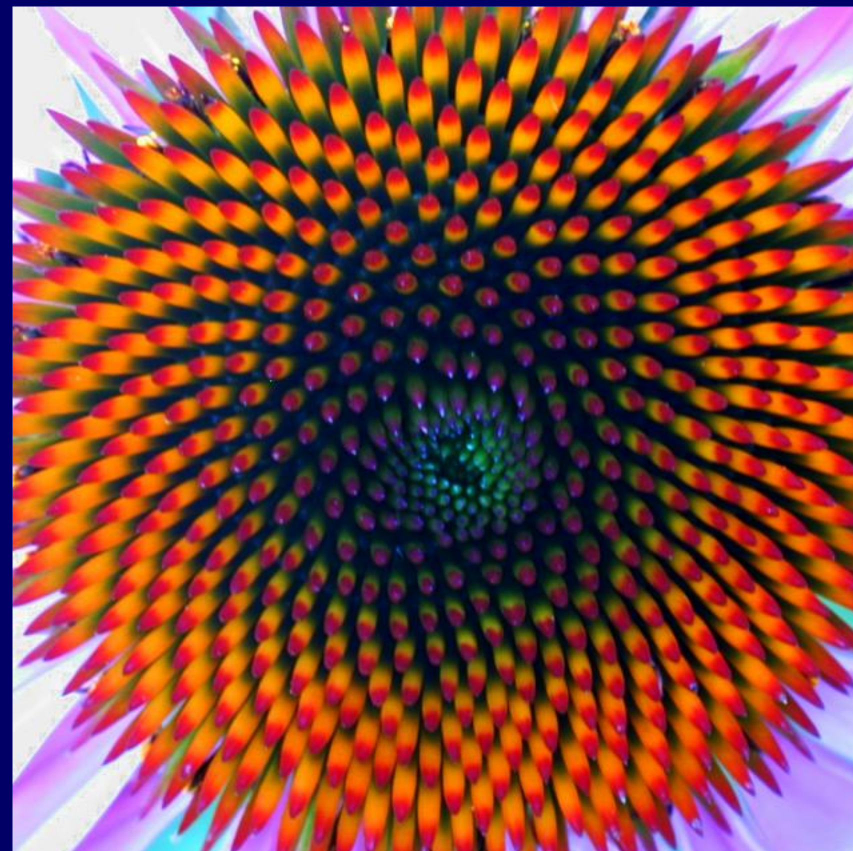
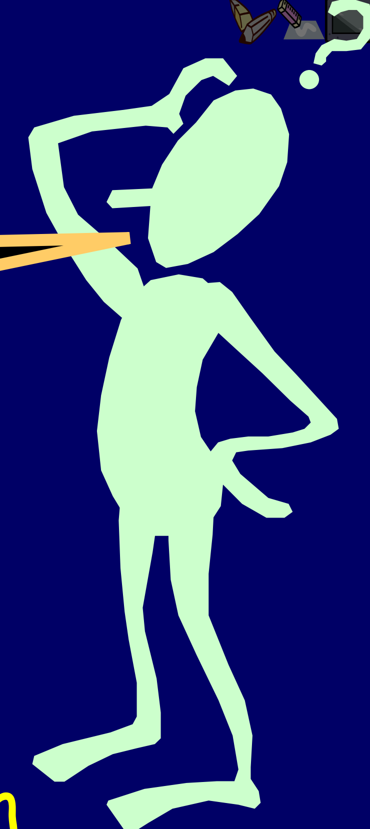


Fig. 12.4. Phyllotaxis





Golden Ratio: the divine proportion

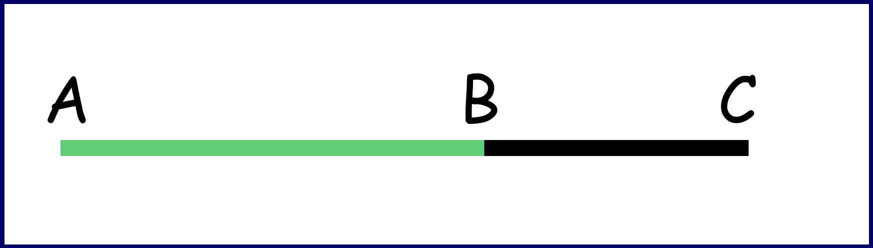
ϕ = 1.6180339887498948482045...

"Phi" is named after the Greek sculptor <u>Phi</u>dias

Definition of ϕ (Euclid)

Ratio obtained when you divide a line segment into two unequal parts such that the ratio of the whole to the larger part is the same as the ratio of the larger to the smaller.

$$\phi = \frac{AC}{AB} = \frac{AB}{BC}$$



$$\phi^2 = \frac{AC}{BC}$$

$$\phi^2 - \phi = \frac{AC}{BC} - \frac{AB}{BC} = \frac{BC}{BC} = 1$$

$$\boxed{\phi^2 - \phi - 1 = 0}$$

Expanding Recursively

$$\phi = 1 + \frac{1}{\phi}$$

$$= 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

Continued Fraction Representation

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}}}$$

Remember?

We already saw the convergents of this CF

[1,1,1,1,1,1,1,1,1,1,1,…]

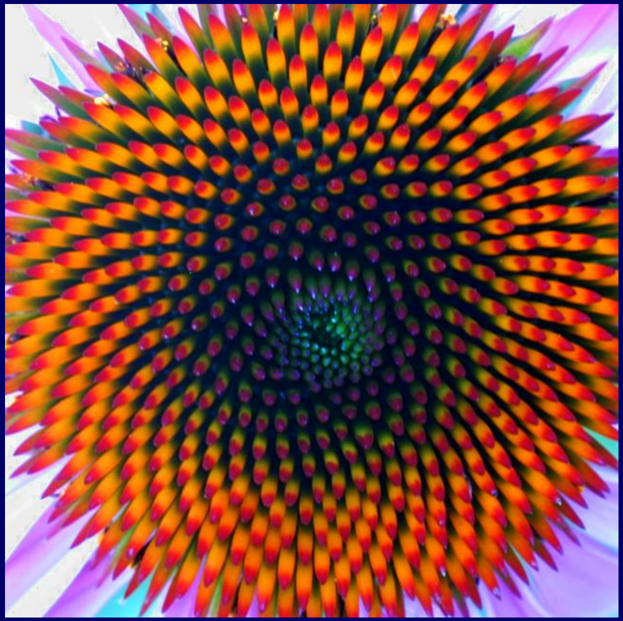
are of the form

Fib(n+1)/Fib(n)

Hence: $$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2}$$

Continued Fraction Representation

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\ldots}}}}}}}}}$$



1,1,2,3,5,8,13,21,34,55,....

2/1	=	2
3/2	=	1.5
5/3	=	1.666...
8/5	=	1.6
13/8	=	1.625
21/13	=	1.6153846...
34/21	=	1.61904...

ϕ = 1.61803398874989484820045

Continued fraction representation of a standard fraction

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \frac{1}{3 + \frac{2}{9}} 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}}$$

e.g., 67/29 = 2 with remainder 9/29
= 2 + 1/ (29/9)

A Representational Correspondence

$$\frac{67}{29} = 2 + \frac{1}{\frac{29}{9}} = 2 + \frac{1}{3 + \frac{2}{9}} \; 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}}$$

Euclid(67,29)	67 div 29 = 2
Euclid(29,9)	29 div 9 = 3
Euclid(9,2)	9 div 2 = 4
Euclid(2,1)	2 div 1 = 2
Euclid(1,0)	

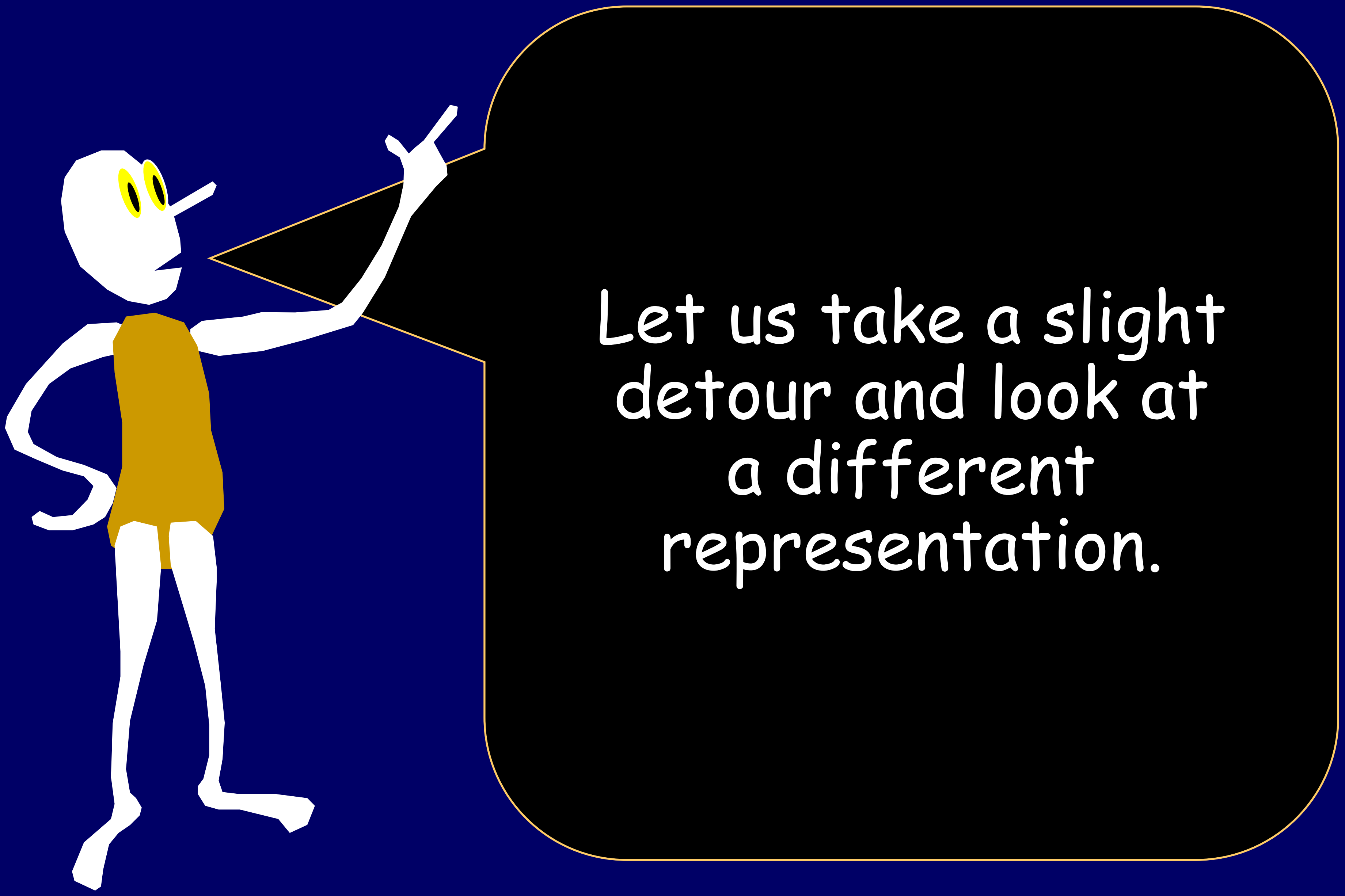
Euclid's GCD = Continued Fractions

$$\frac{A}{B} = \left\lfloor \frac{A}{B} \right\rfloor + \frac{1}{\frac{B}{A \bmod B}}$$

Euclid(A,B) = Euclid(B, A mod B)

Stop when B=0

Theorem: All fractions have finite continuous fraction expansions


Let us take a slight detour and look at a different representation.

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

Example: $f_5 = 5$

Sequences That Sum To n

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Example: $f_5 = 5$

4 = 2 + 2

2 + 1 + 1

1 + 2 + 1

1 + 1 + 2

1 + 1 + 1 + 1

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

f_1

f_3

f_2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$f_1 = 1$

0 = the empty sum

$f_2 = 1$

1 = 1

$f_3 = 2$

2 = 1 + 1

2

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

Sequences That Sum To n

Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

# of sequences beginning with a 1

# of sequences beginning with a 2

Fibonacci Numbers Again

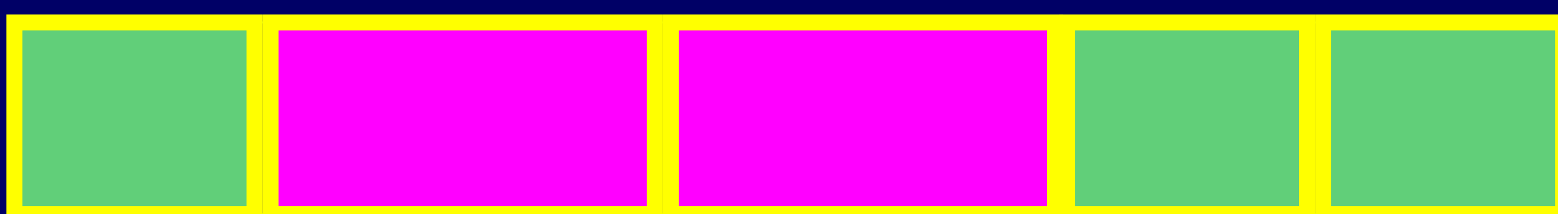
Let f_{n+1} be the number of different sequences of 1's and 2's that sum to n.

$$f_{n+1} = f_n + f_{n-1}$$

$$f_1 = 1 \qquad f_2 = 1$$

Visual Representation: Tiling

Let f_{n+1} be the number of different ways to tile a $1 \times n$ strip with squares and dominoes.



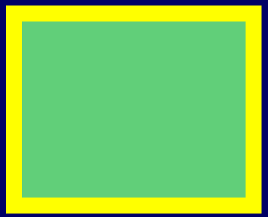
Visual Representation: Tiling

Let f_{n+1} be the number of different ways to tile a $1 \times n$ strip with squares and dominoes.

Visual Representation: Tiling

1 way to tile a strip of length 0

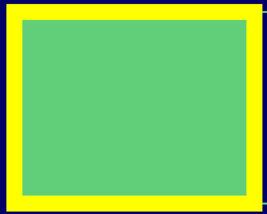
1 way to tile a strip of length 1:



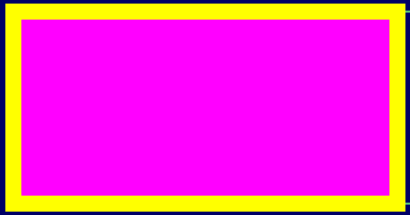
2 ways to tile a strip of length 2:

$$f_{n+1} = f_n + f_{n-1}$$

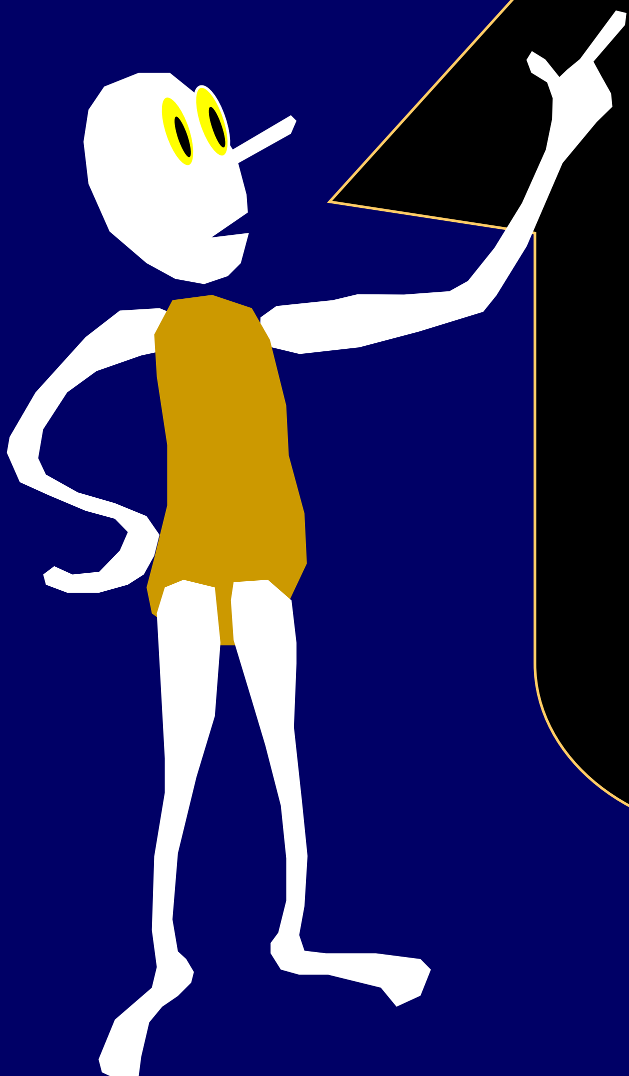
f_{n+1} is number of ways to tile length n.



f_n tilings that start with a square.



f_{n-1} tilings that start with a domino.



Let's use this visual representation to prove a couple of Fibonacci identities.

Fibonacci Identities

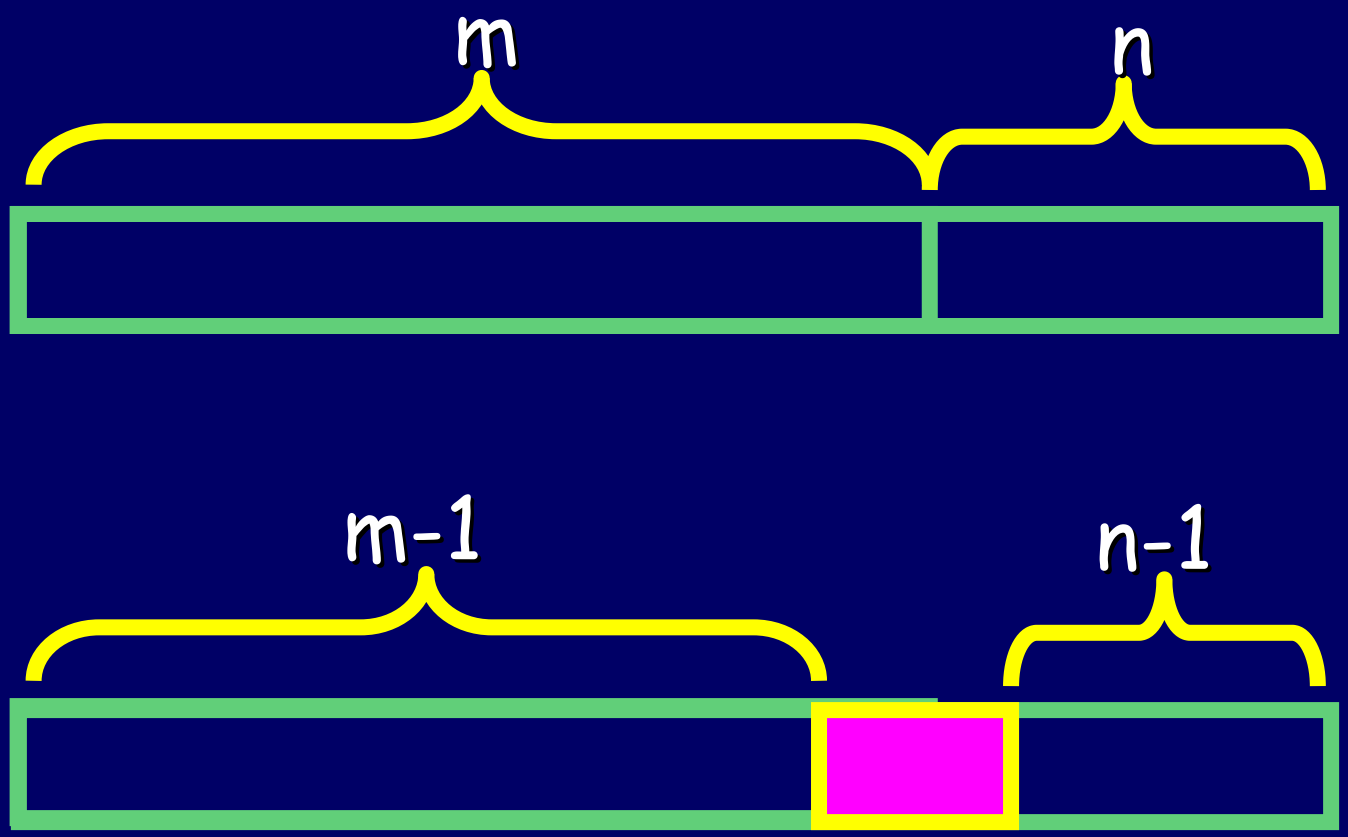
Some examples:

$$F_{2n} = F_1 + F_3 + F_5 + \ldots + F_{2n-1}$$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

$$F_{m+n+1} = F_{m+1} F_{n+1} + F_m F_n$$



$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

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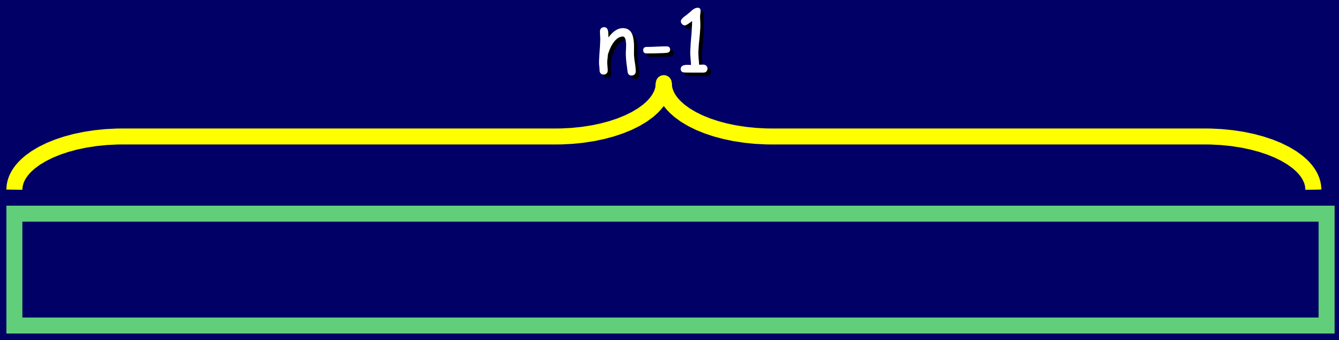
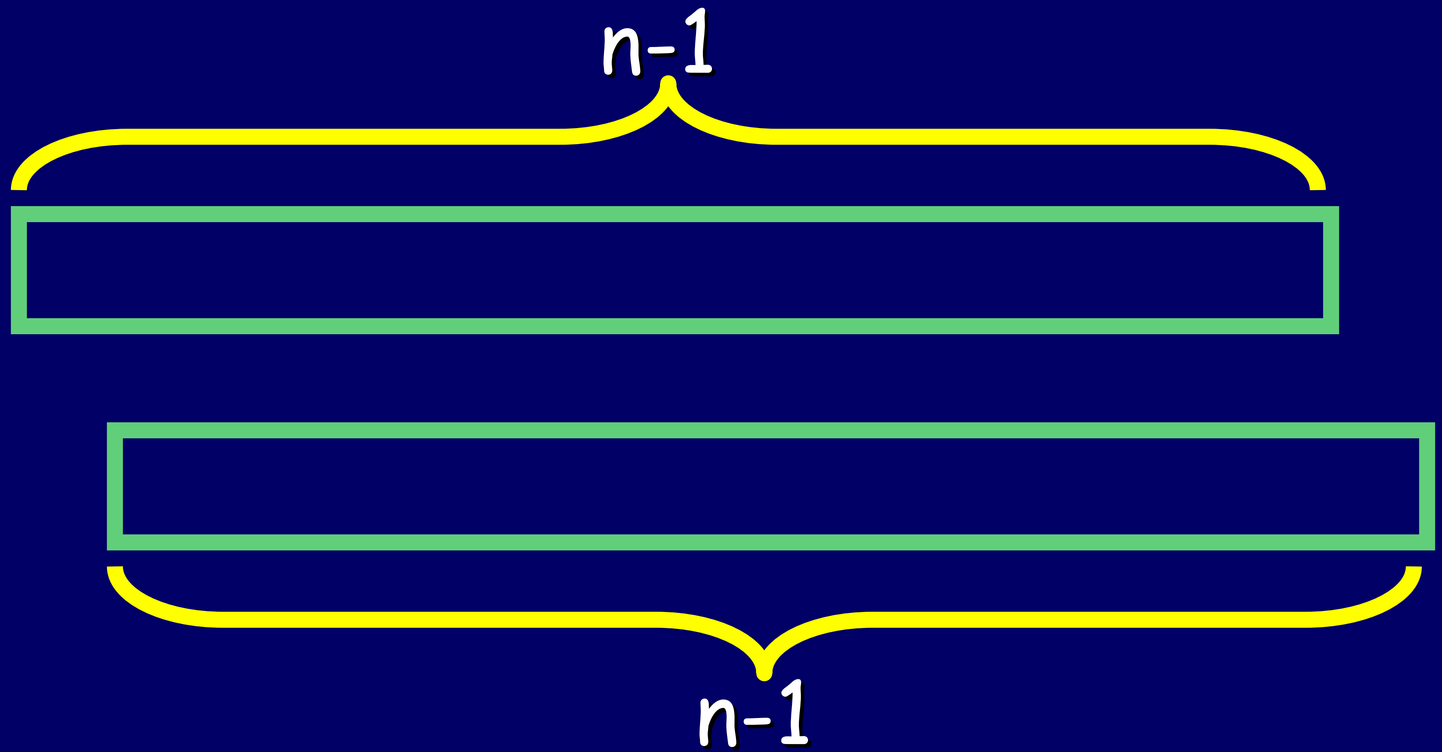
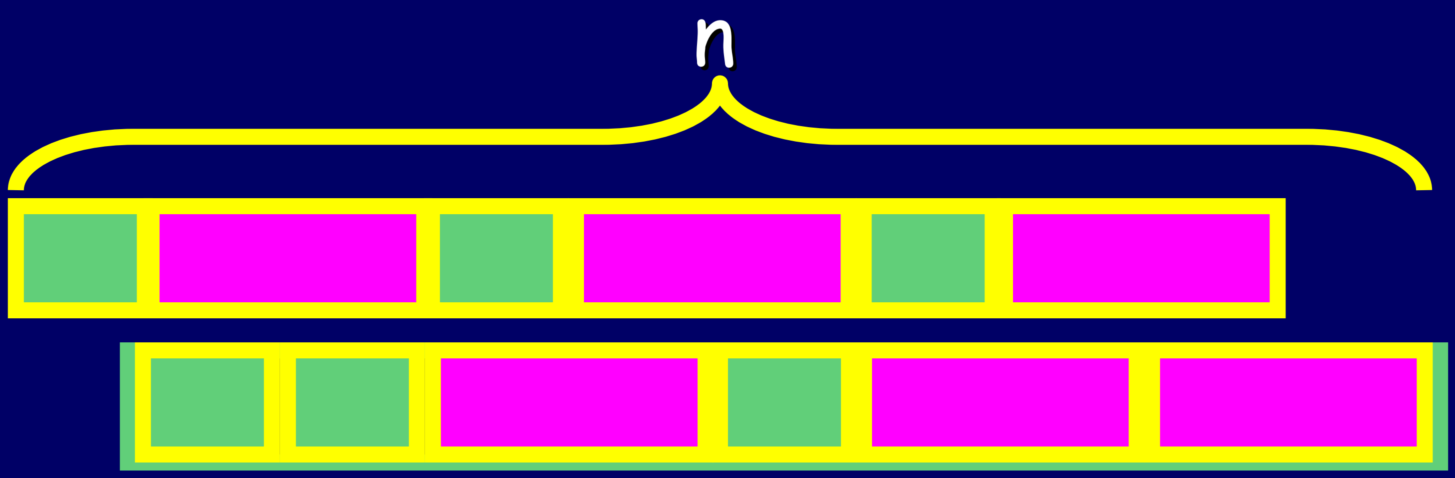


F_n tilings of a strip of length n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$

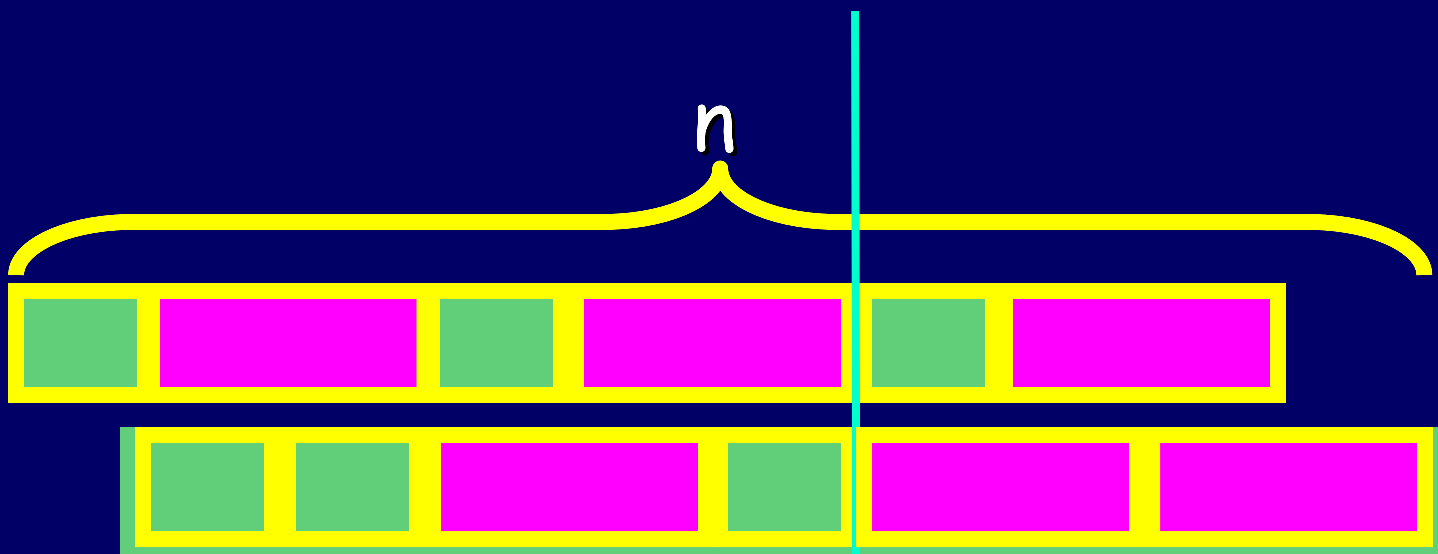


$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



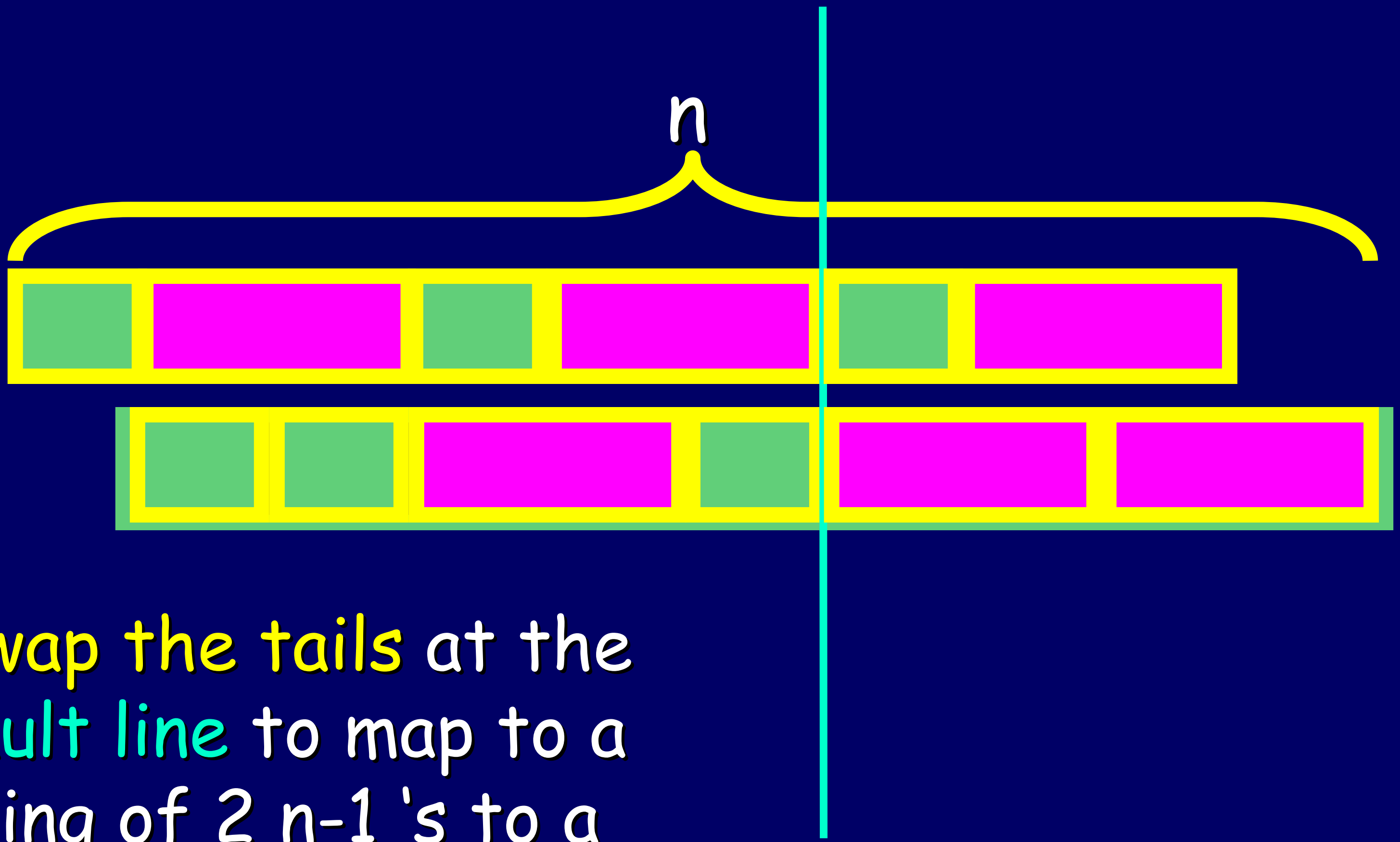
$(F_n)^2$ tilings of two strips of size n-1

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



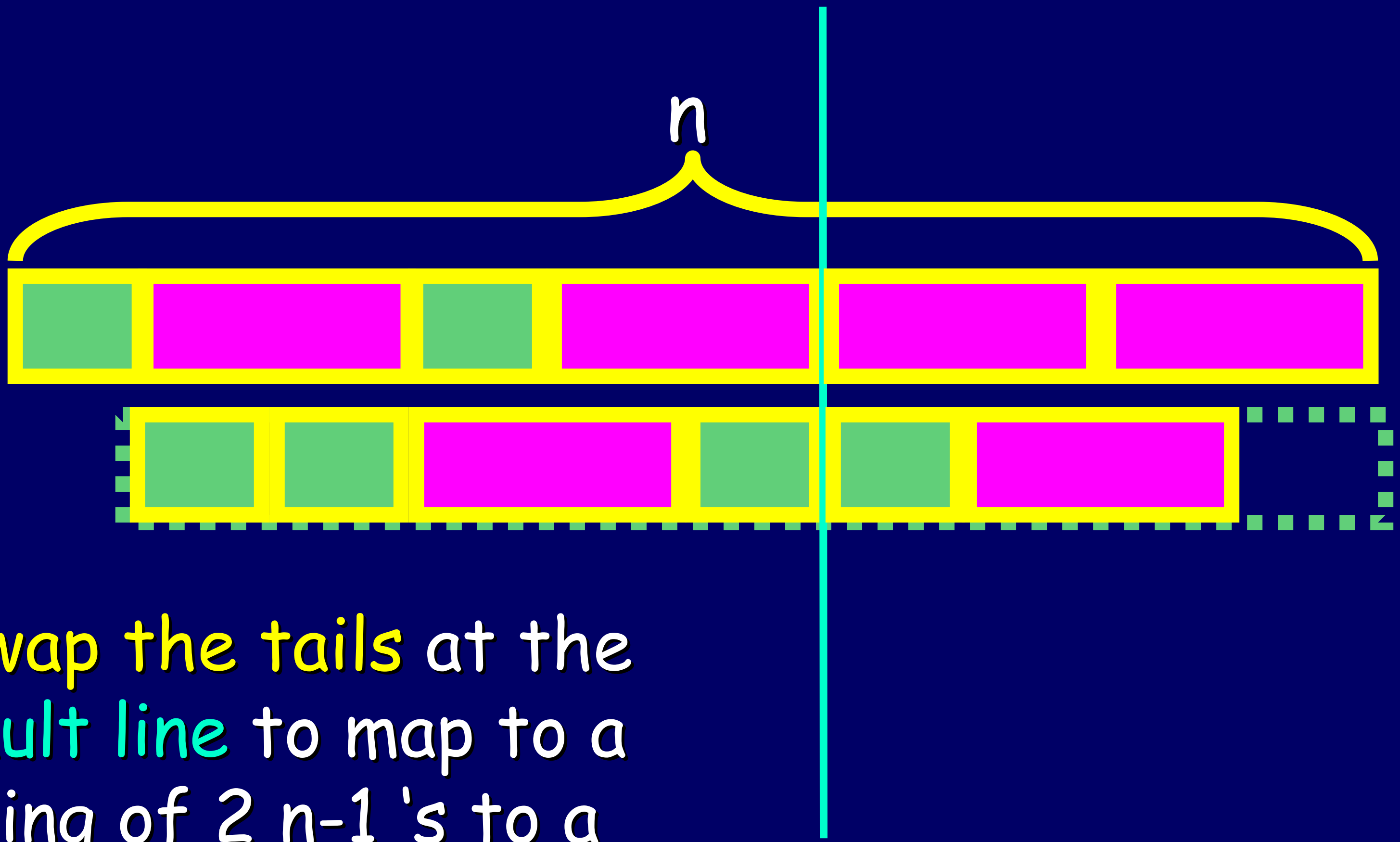
Draw a vertical "fault line" at the rightmost position (<n) possible without cutting any dominoes

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



Swap the tails at the fault line to map to a tiling of 2 n-1 's to a tiling of an n-2 and an n.

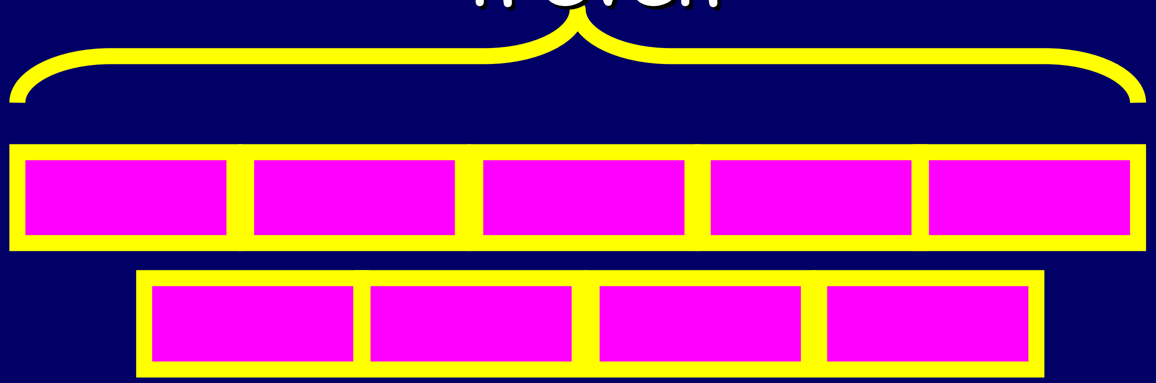
$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^n$$



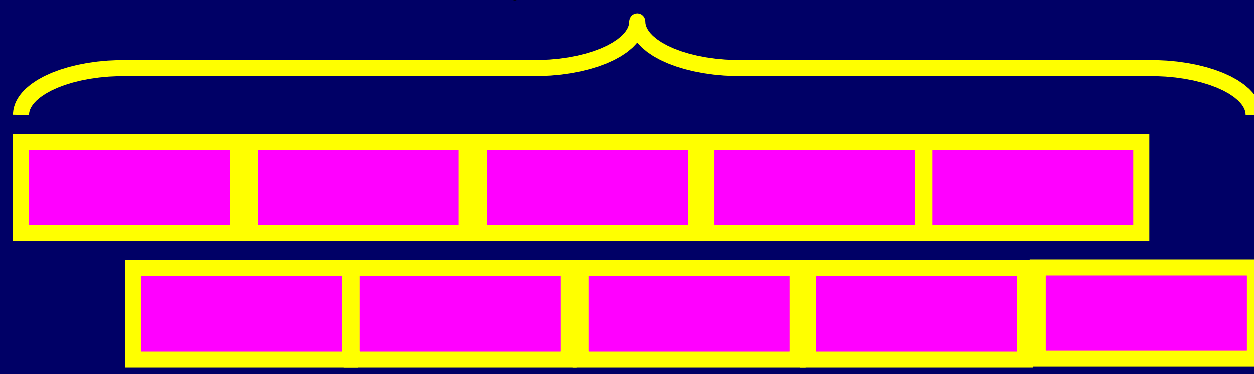
Swap the tails at the fault line to map to a tiling of 2 n-1 's to a tiling of an n-2 and an n.

$$(F_n)^2 = F_{n-1} F_{n+1} + (-1)^{n-1}$$

n even



n odd



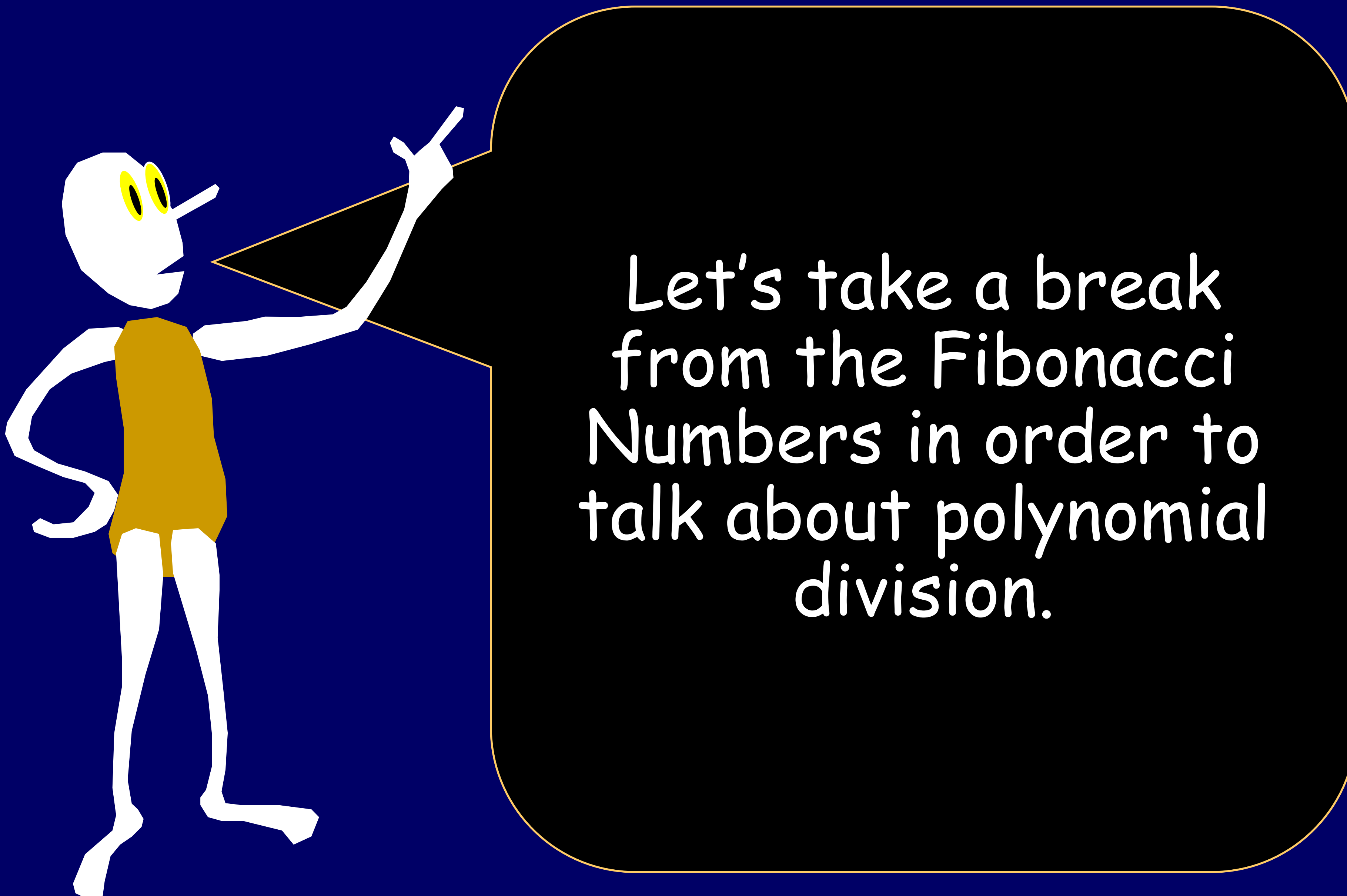
More random facts

The product of any four consecutive Fibonacci numbers is the area of a Pythagorean triangle.

The sequence of final digits in Fibonacci numbers repeats in cycles of 60. The last two digits repeat in 300, the last three in 1500, the last four in 15,000, etc.

Useful to convert miles to kilometers.

The Fibonacci Quarterly


Let's take a break from the Fibonacci Numbers in order to talk about polynomial division.

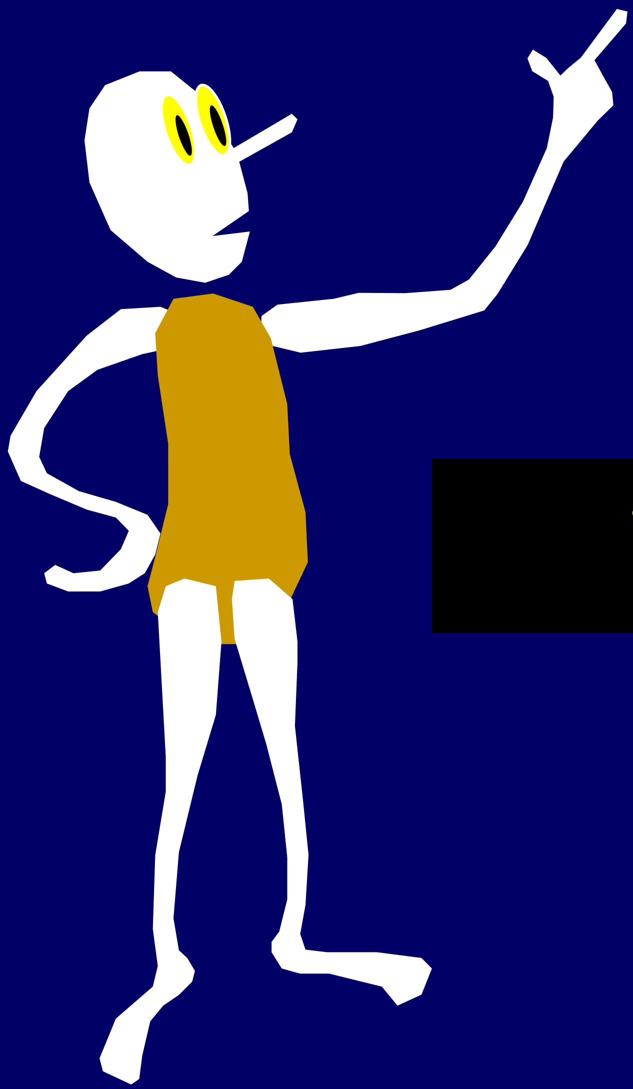
How to divide polynomials?

$$\frac{1}{1 - X} \;?$$

$$
\begin{array}{r|l}
 & 1 + X + X^2 \\ \hline
1 - X & 1 \\
 & -(1 - X) \\ \hline
 & \quad X \\
 & \quad -(X - X^2) \\ \hline
 & \qquad X^2 \\
 & \qquad -(X^2 - X^3) \\ \hline
 & \qquad\quad X^3 \quad \ldots
\end{array}
$$

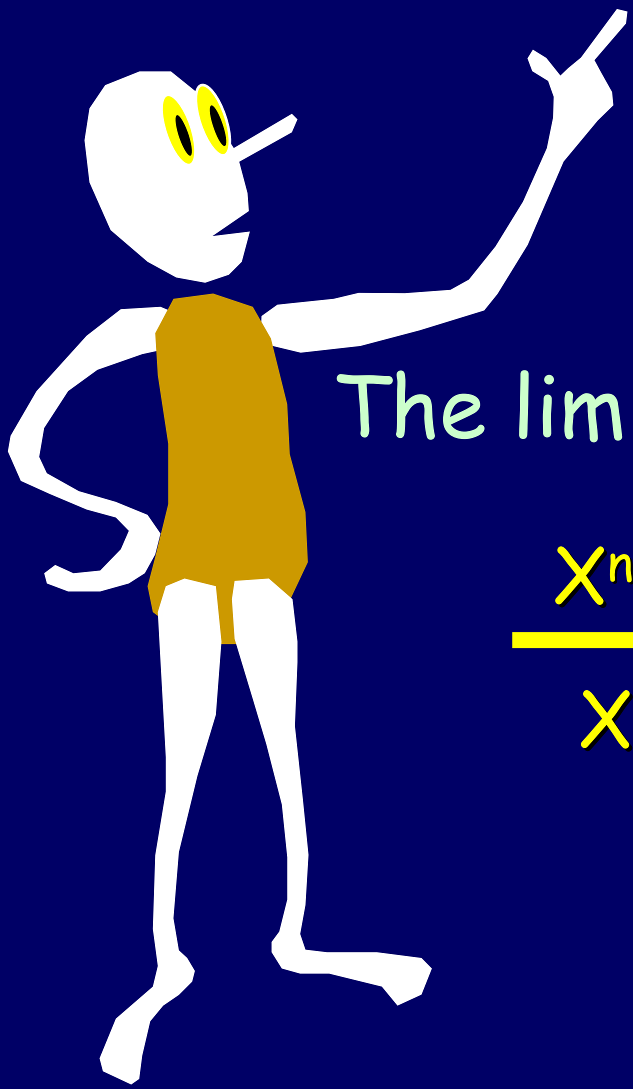
$$= \quad 1 + X + X^2 + X^3 + X^4 + X^5 + X^6 + X^7 + \ldots$$

$$1 + X^{1} + X^{2} + X^{3} + \ldots + X^{n-1} + X^{n} = \frac{X^{n+1} - 1}{X - 1}$$



The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$



The limit as n goes to infinity of

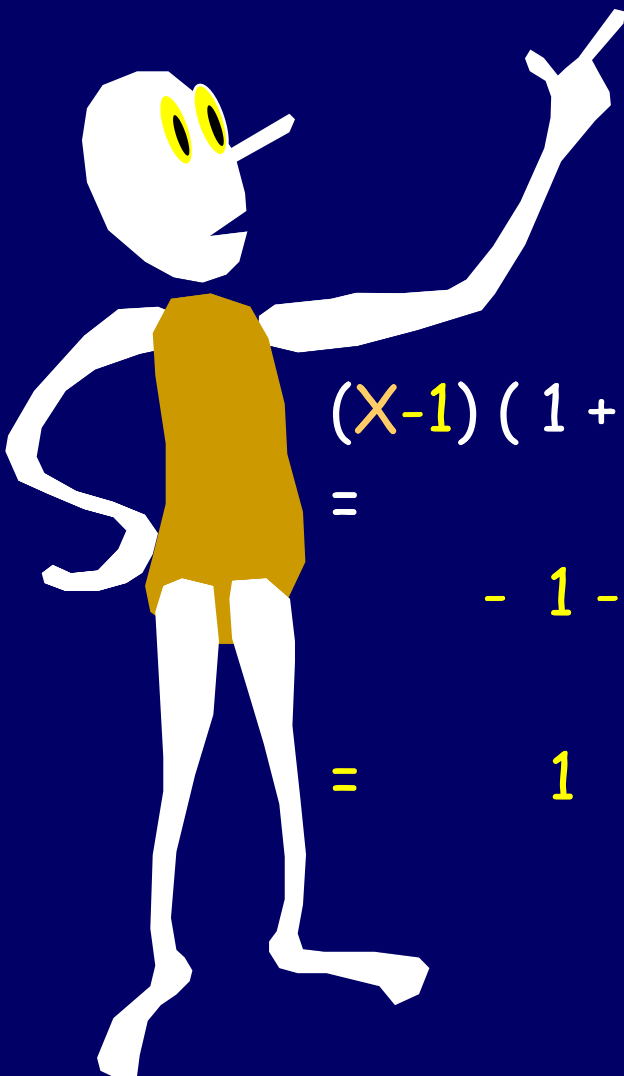
$$\frac{X^{n+1} - 1}{X - 1} = \frac{-1}{X - 1}$$

$$= \frac{1}{1 - X}$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$

The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



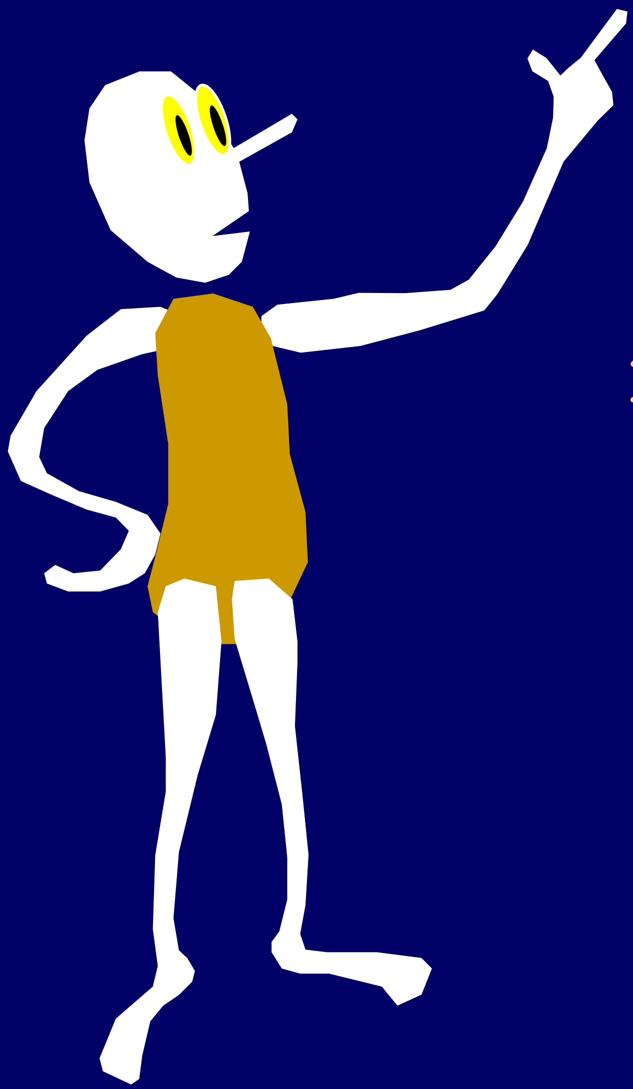
$$(X-1)\,(1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots)$$

$$= \quad X^1 + X^2 + X^3 + \ldots \quad + X^n + X^{n+1} + \ldots$$

$$- 1 - X^1 - X^2 - X^3 - \ldots - X^{n-1} - X^n - X^{n+1} - \ldots$$

$$= \quad 1$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots\ldots = \frac{1}{1 - X}$$



$$
\begin{array}{r|l}
 & 1 + X + X^2 + \ldots \\
\hline
1 - X & 1 \\
 & \underline{-(1 - X)} \\
 & \quad X \\
 & \quad \underline{-(X - X^2)} \\
 & \qquad X^2 \\
 & \qquad \underline{-(X^2 - X^3)} \\
 & \qquad\quad X^3 \\
 & \qquad\qquad \ldots
\end{array}
$$

Something a bit more complicated

$$\frac{X}{1 - X - X^2}$$

Quotient: $X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6$

Divisor: $1 - X - X^2$

X

$-(X - X^2 - X^3)$

$X^2 + X^3$

$-(X^2 - X^3 - X^4)$

$2X^3 + X^4$

$-(2X^3 - 2X^4 - 2X^5)$

$3X^4 + 2X^5$

$-(3X^4 - 3X^5 - 3X^6)$

$5X^5 + 3X^6$

$-(5X^5 - 5X^6 - 5X^7)$

$8X^6 + 5X^7$

$-(8X^6 - 8X^7 - 8X^8)$

Hence

$$\frac{X}{1 - X - X^2}$$

$$= 0 \times 1 + 1\,X^1 + 1\,X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \ldots$$

$$= F_0\,1 + F_1\,X^1 + F_2\,X^2 + F_3\,X^3 + F_4\,X^4 + F_5\,X^5 + F_6\,X^6 + \ldots$$

Going the Other Way

$F_0 = 0, F_1 = 1$

$(1 - X - X^2) \times$

$(F_0 1 + F_1 X^1 + F_2 X^2 + \ldots + F_{n-2} X^{n-2} + F_{n-1} X^{n-1} + F_n X^n + \ldots$

Going the Other Way

$(1 - X - X^2) \times$

$(F_0 1 + F_1 X^1 + F_2 X^2 + \ldots + F_{n-2} X^{n-2} + F_{n-1} X^{n-1} + F_n X^n + \ldots$

$= (F_0 1 + F_1 X^1 + F_2 X^2 + \ldots + F_{n-2} X^{n-2} + F_{n-1} X^{n-1} + F_n X^n + \ldots$

$- F_0 X^1 - F_1 X^2 - \ldots - F_{n-3} X^{n-2} - F_{n-2} X^{n-1} - F_{n-1} X^n - \ldots$

$- F_0 X^2 - \ldots - F_{n-4} X^{n-2} - F_{n-3} X^{n-1} - F_{n-2} X^n - \ldots$

$= F_0 1 + (F_1 - F_0) X^1$

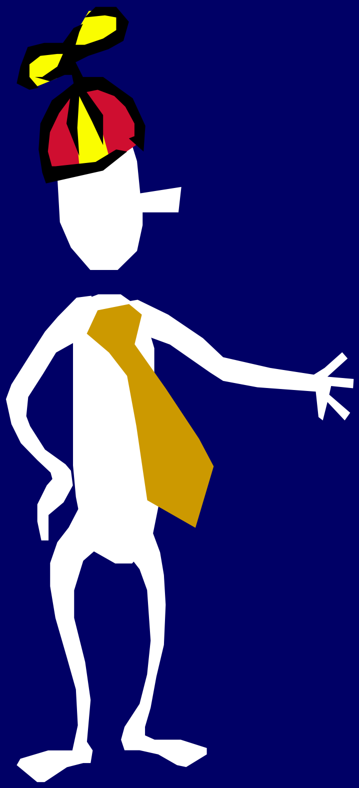
$= X$

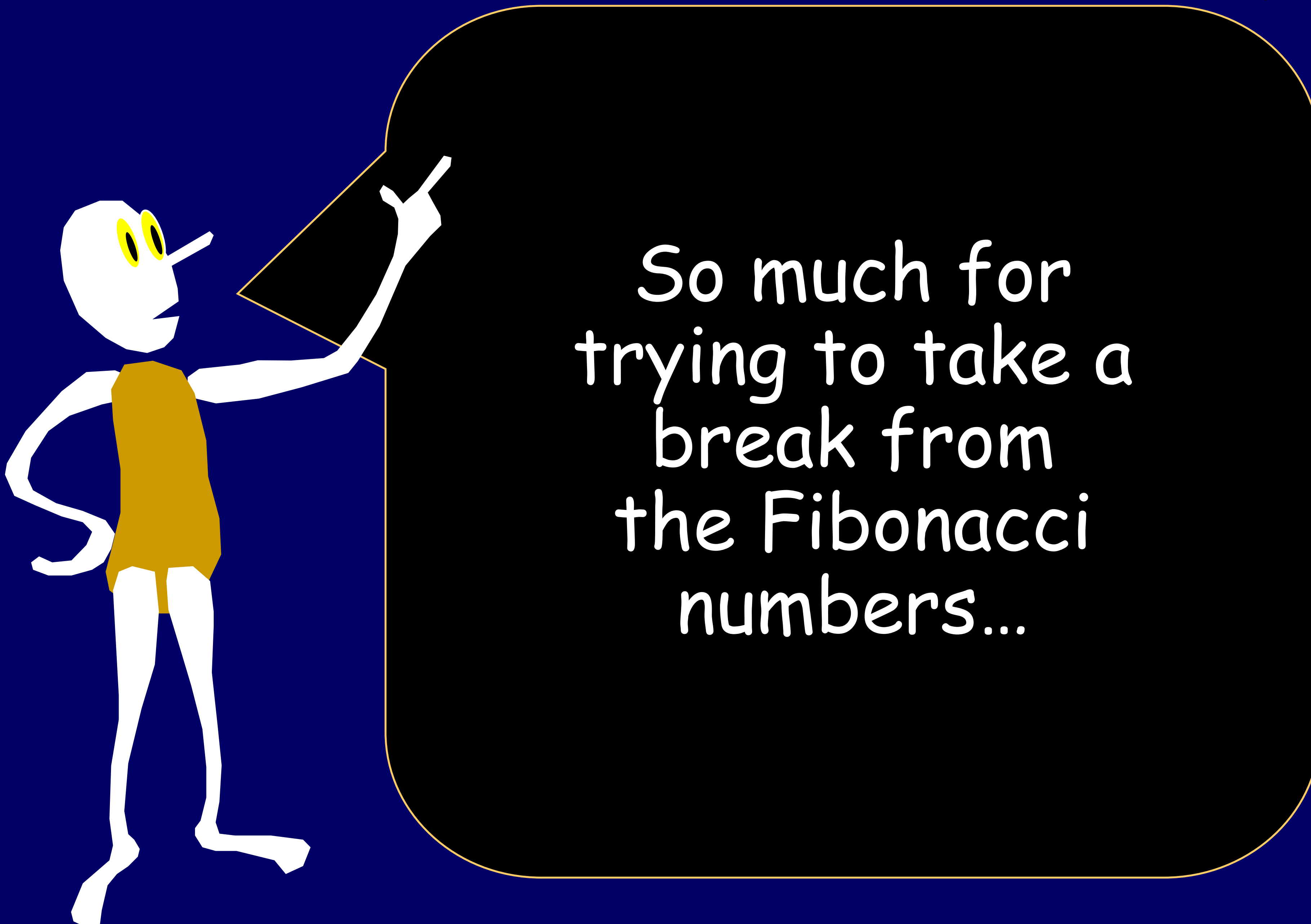
$F_0 = 0, F_1 = 1$

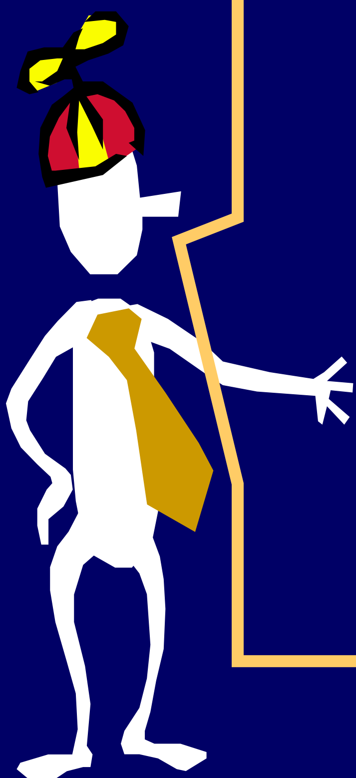
Thus

$$F_0\, 1 + F_1\, X^1 + F_2\, X^2 + \ldots + F_{n-1}\, X^{n-1} + F_n\, X^n + \ldots$$

$$= \frac{X}{1 - X - X^2}$$




So much for trying to take a break from the Fibonacci numbers…

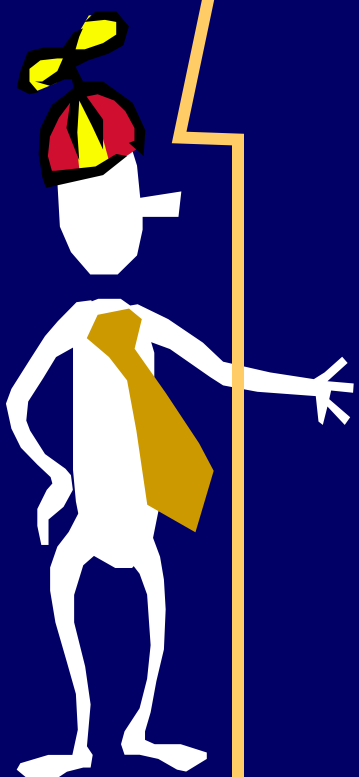


What is the Power Series Expansion of $x/(1-x-x^2)$?

What does this look like when we expand it as an infinite sum?

ϕ is not of $x^2 - x - 1 = 0$ $\frac{1 \pm \sqrt{5}}{2}$

$-\frac{1}{\phi}$



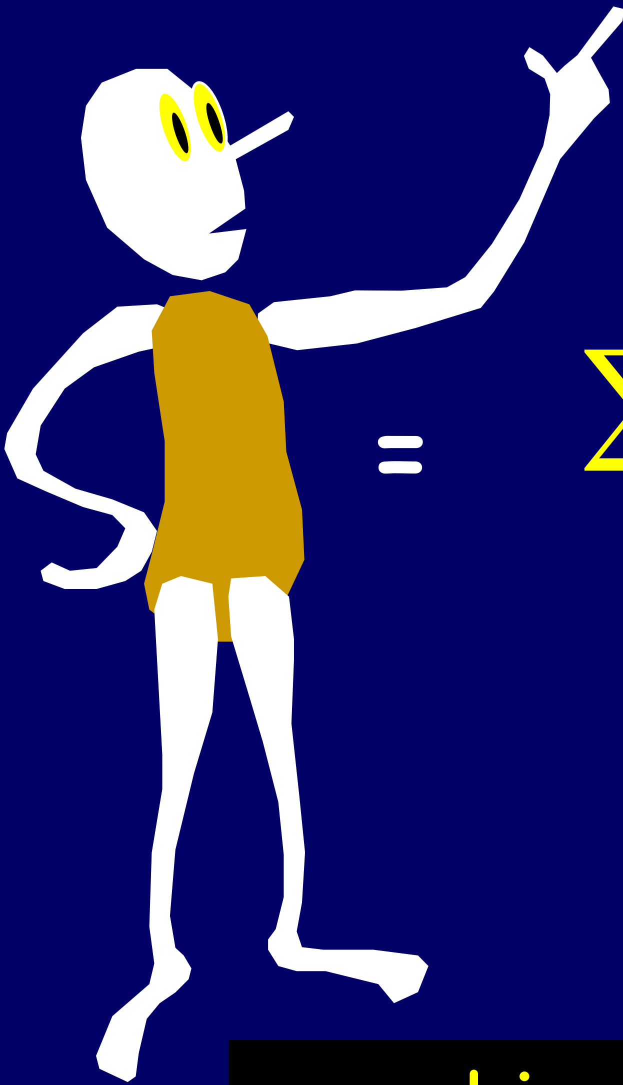
Since the bottom is quadratic we can factor it.

$$X / (1 - X - X^2) =$$

$$X / (1 - \phi X)(1 - (-\phi)^{-1} X)$$

$$\text{where } \phi = \frac{1 + \sqrt{5}}{2}$$

"The Golden Ratio"



$$\frac{X}{(1 - \phi X)(1 - (-\phi)^{-1} X)}$$

$$= \sum_{n=0..\infty} \; ? \; X^n$$

Linear factors on the bottom

$$\left(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\right)\left(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\right) =$$

$$\frac{1}{(1-ax)^2} = \frac{1}{a}\frac{d}{dx}\frac{1}{1-aX}$$

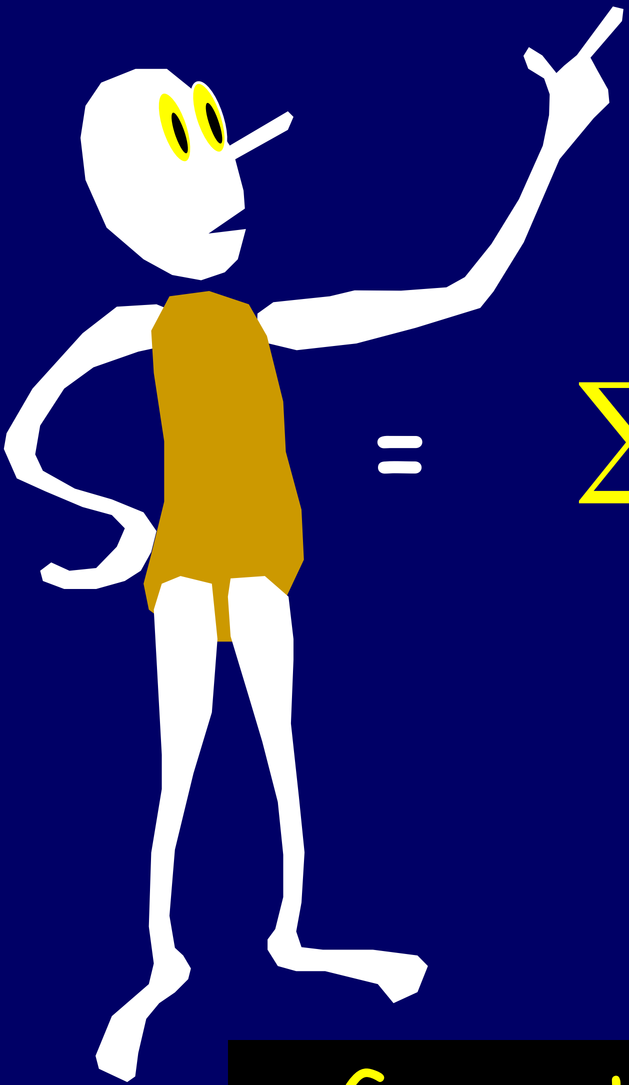
$$= \frac{1}{(1 - aX)(1-bX)}$$

na^n coeff. of X^{n-1}

$a \neq b$.

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

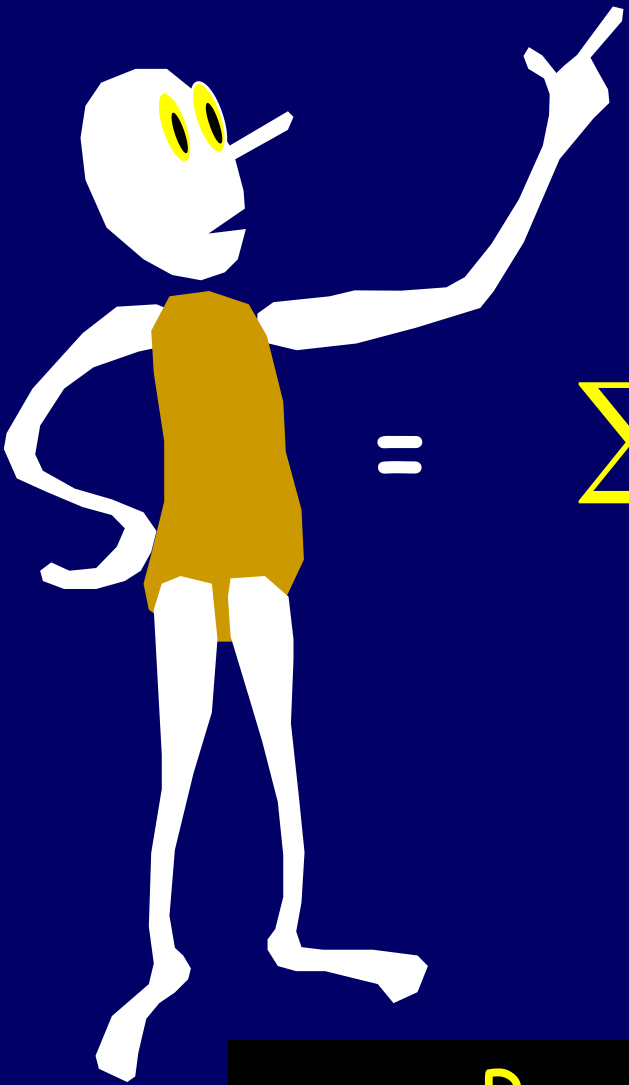
Geometric Series (Quadratic Form)



$$\frac{1}{(1 - \phi X)(1- (-\phi^{-1}X)}$$

$$= \sum_{n=0..\infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^n$$

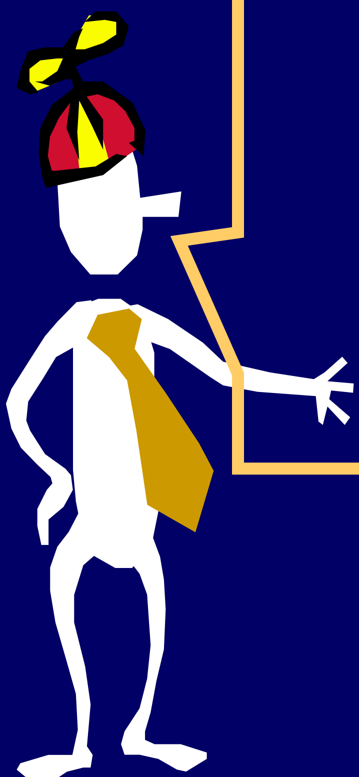
Geometric Series (Quadratic Form)



$$\frac{X}{(1 - \phi X)(1 - (-\phi^{-1} X)} = \sum_{n=0..\infty} \frac{\phi^{n+1} - (-\phi^{-1})^{n+1}}{\sqrt{5}} X^{n+1}$$

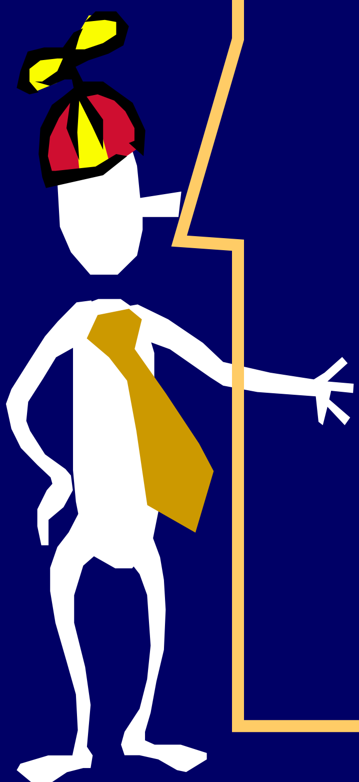
Power Series Expansion of F

$$\frac{x}{1-x-x^2} = F_0x^0 + F_1x^1 + F_2x^2 + F_3x^3 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$



$$\frac{x}{1-x-x^2} = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}}\left(\phi^i - \left(-\frac{1}{\phi}\right)^i\right)x^i$$

Leonhard Euler (1765)
J. P. M. Binet (1843)
A de Moivre (1730)



The i^{th} Fibonacci number is:

$$\frac{1}{\sqrt{5}}\left(\phi^{i}-\left(-\frac{1}{\phi}\right)^{i}\right)$$

$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}} - \frac{\left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$

≈ 0 for large n

Less than .277

$$F_n = \text{closest integer to } \frac{\phi^n}{\sqrt{5}} = \left[\frac{\phi^n}{\sqrt{5}}\right]$$

$$F_n \approx \frac{\phi^n}{\sqrt{5}}$$

$$\frac{F_n}{F_{n-1}} = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} = \frac{\phi^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}} + \frac{-\left(\frac{-1}{\phi}\right)^n}{\phi^{n-1} - \left(\frac{-1}{\phi}\right)^{n-1}}$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi$$

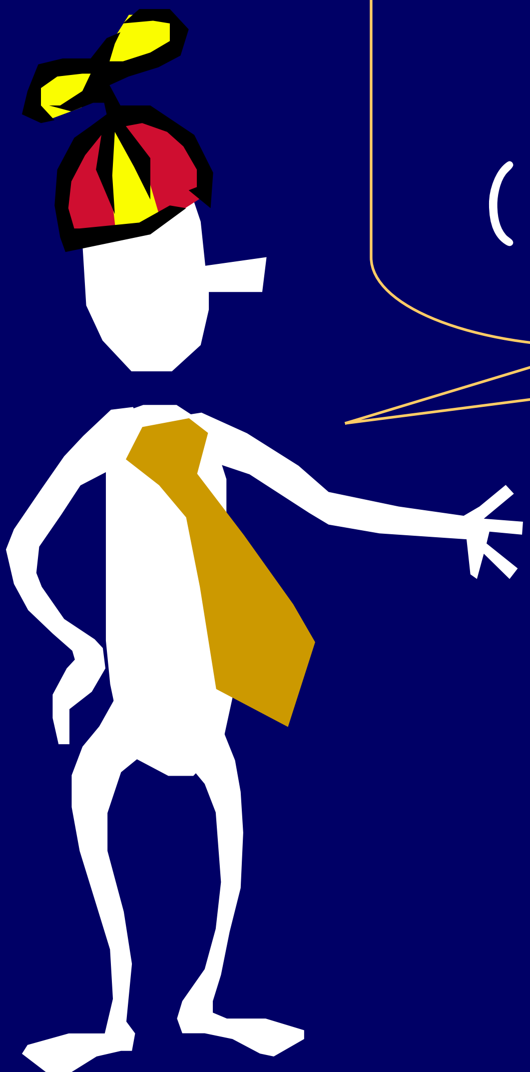
What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots)^n ?$$

Each path in the choice tree for the cross terms has n choices of exponent $e_1, e_2, \ldots, e_n \geq 0$. Each exponent can be any natural number.

Coefficient of X^k is the number of non-negative solutions to:

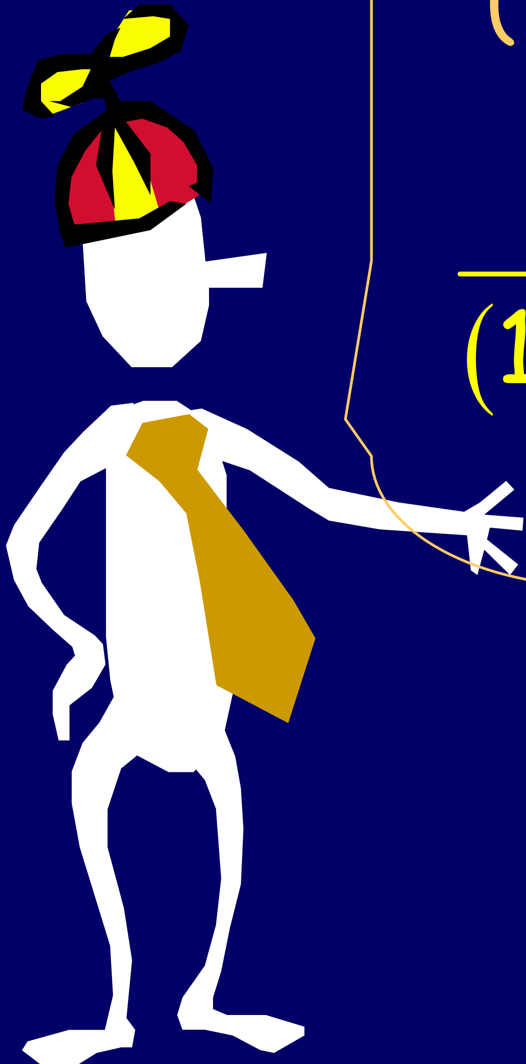
$$e_1 + e_2 + \ldots + e_n = k$$



What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots)^n ?$$

$$\binom{n + k - 1}{n - 1}$$



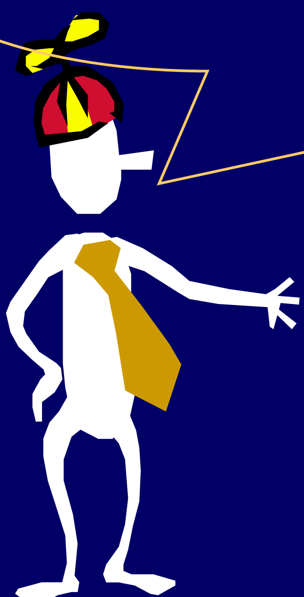
$$(1 + x + x^2 + x^3 + x^4 + \dots)^n =$$

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

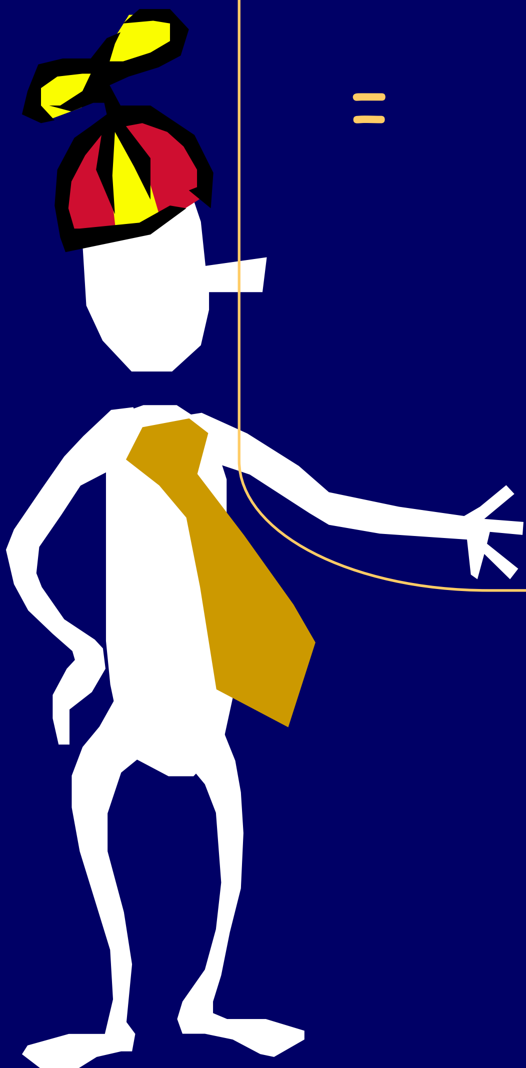
What is the coefficient of X^k in the expansion of:

$$(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots)\,(1 + X + X^2 + X^3 + \ldots)$$

$$= (a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots) \,/\, (1 - X) \qquad ?$$



$$a_0 + a_1 + a_2 + \ldots + a_k$$

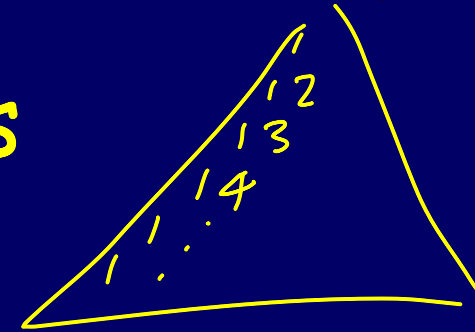


$$(a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots) \;/\; (1 - X)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{i=k} a_i\right)X^k$$

Some simple power series

Al-Karaji's Identities



Zero_Ave = $1/(1-X)$;

First_Ave = $1/(1-X)^2$;

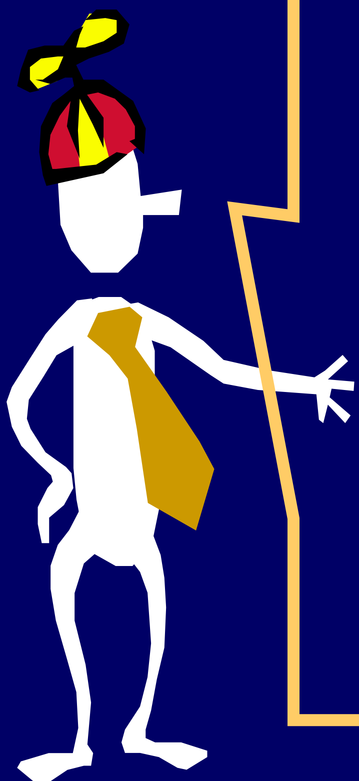
Second_Ave = $1/(1-X)^3$;

Output =

$$1/(1-X)^2 + 2X/(1-X)^3$$

$$= (1-X)/(1-X)^3 + 2X/(1-X)^3$$

$$= (1+X)/(1-X)^3$$



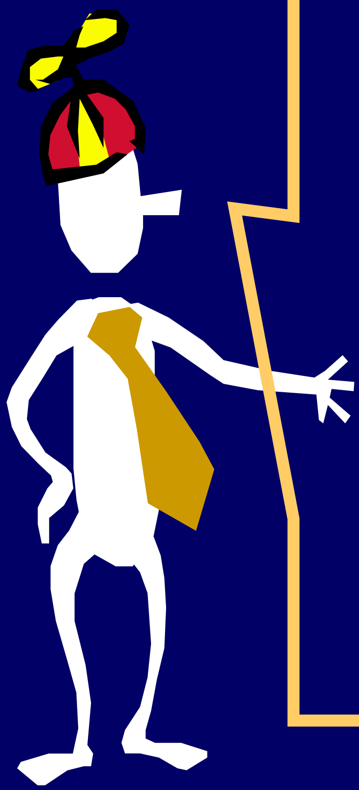
$$(1+X)/(1-X)^3$$

outputs $\langle 1, 4, 9, .. \rangle$

$$X(1+X)/(1-X)^3$$

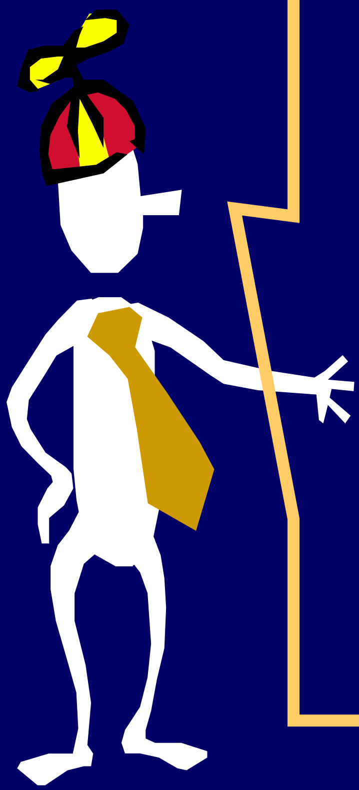
outputs $\langle 0, 1, 4, 9, .. \rangle$

The k^{th} entry is k^2



$$X(1+X)/(1-X)^3 = \sum k^2 X^k$$

What does $X(1+X)/(1-X)^4$ do?



$X(1+X)/(1-X)^4$ expands to :

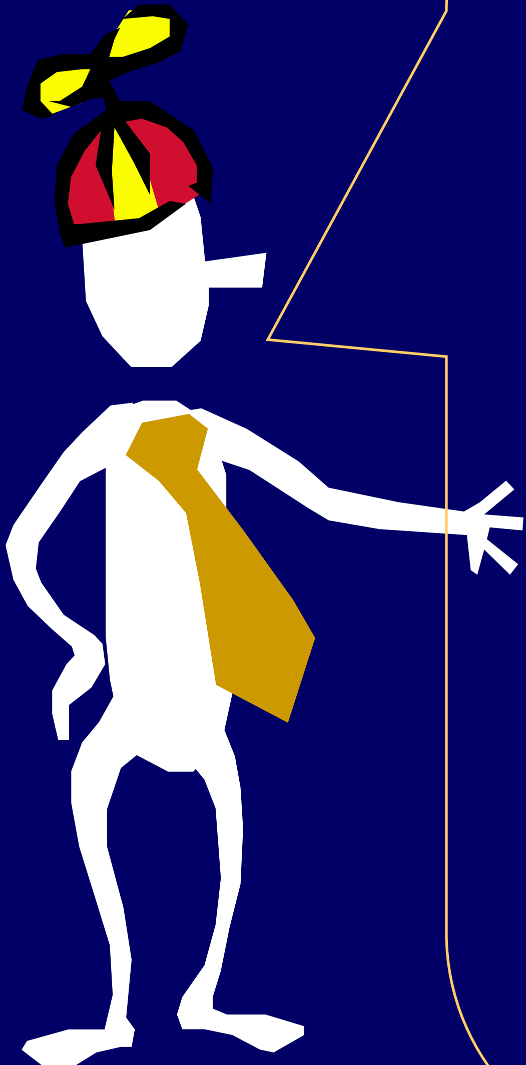
$$\sum S_k X^k$$

where S_k is the sum of the first k squares

Aha! Thus, if there is an alternative interpretation of the k^{th} coefficient of

$$X(1+X)/(1-X)^4$$

we would have a new way to get a formula for the sum of the first k squares.



Using pirates and gold we found that:

$$\frac{1}{(1-X)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} X^k$$

THUS:

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty} \binom{k+3}{3} X^k$$

Coefficient of X^k in $P_V = (X^2+X)(1-X)^{-4}$ is the sum of the first k squares:

$$\frac{X^2+X}{(1-X)^4} = (X^2+X)\sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

$$= \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\frac{1}{(1-X)^4} = \sum_{k=0}^{\infty}\binom{k+3}{3}X^k$$

Polynomials give us closed form expressions

$$\frac{X^2+X}{(1-X)^4} = \sum_{k=0}^{\infty}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)X^k$$

$$\sum_{i=0}^{i=n} i^2 = \binom{n+2}{3}+\binom{n+1}{3}$$

REFERENCES

Coxeter, H. S. M. ``The Golden Section, Phyllotaxis, and Wythoff's Game.'' *Scripta Mathematica* **19**, 135-143, 1953.

"Recounting Fibonacci and Lucas Identities" by Arthur T. Benjamin and Jennifer J. Quinn, College Mathematics Journal, Vol. 30(5): 359--366, 1999.

Study Bee

Fibonacci Numbers

- Arise everywhere
- Visual Representations
- Fibonacci Identities

Polynomials

- The infinite geometric series
- Division of polynomials
- Representation of Fibonacci numbers as coefficients of polynomials.

Generating Functions and Power Series

- Simple operations (add, multiply)
- Quadratic form of the Geometric Series
- Deriving the closed form for F_n

- Pirates and gold
- Sum of squares once again!