Theorem. Let $G$ be a graph with $n$ nodes and $e$ edges. Then the following are equivalent:

1. $G$ is a tree (connected and acyclic).
2. Every two nodes of $G$ are joined by a unique path.
3. $G$ is connected and $n = e + 1$.
4. $G$ is acyclic and $n = e + 1$.
5. $G$ is acyclic and if any two nonadjacent points are joined by a line, the resulting graph has exactly one cycle.

Proof. We need to show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$.

1 $\Rightarrow$ 2: If $G$ is a tree then every two nodes are joined by a unique path. Suppose that there are two paths $P_1$ and $P_2$ between node $u$ and node $v$. Tracing the two paths simultaneously from $u$ to $v$, let $w$ be the first point that is on both paths, but for which the successor points are on different paths. Also, let $x$ be the next point after $w$ that is is on both paths. Then the paths between $w$ and $x$ on $P_1$ and $P_2$ together make a cycle. But this can’t happen if $G$ is acyclic.

2 $\Rightarrow$ 3: If every two nodes of $G$ are joined by a unique path, then $G$ is connected and $n = e + 1$. $G$ is connected since any two nodes are joined by a path. To show $n = e + 1$, we use induction. Assume it’s true for less than $n$ points. Removing any edge from $G$ breaks $G$ into two components, since paths are unique. Suppose the sizes are $n_1$ and $n_2$, with $n_1 + n_2 = n$. By the induction hypothesis, $n_1 = e_1 + 1$ and $n_2 = e_2 + 1$; but then $n = n_1 + n_2 = (e_1 + 1) + (e_2 + 1) = (e_1 + e_2) + 2 = e - 1 + 2 = e + 1$.

3 $\Rightarrow$ 4: If $G$ is connected and $n = e + 1$ then $G$ is acyclic. Suppose $G$ has a cycle of length $k$. Then there are $k$ points and $k$ edges on this cycle. Since $G$ is connected, for each node $v$ not on the cycle, there is a shortest path from $v$ to a node on the cycle. Each such path contains an edge $e_v$ not on any other (since they are shortest paths). Thus, the number of edges is at least $e \geq (n - k) + k = n$, which contradicts the assumption $n = e + 1$.

4 $\Rightarrow$ 5: If $G$ is acyclic and $n = e + 1$ then if any two nonadjacent points are joined by a line, the resulting graph has exactly one cycle. Since $G$ doesn’t have cycles, each component of $G$ is a tree. Suppose there are $k$ components. Thus, $n_i = e_i + 1$ if the $i$-th component has $e_i$ edges and $n_i$ nodes, and therefore $n = e + k$. It follows that $k = 1$, so $G$ is in fact connected, and therefore a tree. For any pair of disconnected nodes $u$ and $v$, there is a unique path between them. Adding the the line $(u, v)$ thus results in a single cycle.

5 $\Rightarrow$ 1: If $G$ is acyclic joining any nonadjacent points results in a unique cycle, then $G$ is a tree. Since joining any pair of nonadjacent points gives a cycle, the points must be connected by a path. Thus $G$ is connected.
Another Proof of Cayley’s Formula

There are several other elegant proofs of Cayley’s formula. Here we’ll give a combinatorial proof that uses induction, and a strengthening of the induction hypothesis.

A collection of trees is, naturally, called a forest. Let $T_{n,k}$ be the number of labeled forests on $\{1, \ldots, n\}$ consisting of $k$ trees. Then $T_{n,1}$ is the number of labeled trees; we want to show that $T_{n,1} = n^{n-2}$.

Suppose we have a forest $F$ with $k$ trees; we can assume that the vertices $\{1, 2, \ldots, k\}$ are in different trees. Suppose that vertex 1 is adjacent to $i$ nodes. Then if we delete vertex 1, the $i$ neighbors together with 2, $\ldots$, $k$ give one vertex each in the components of a forest with $k - 1 + i$ trees. We can reconstruct the original forest by first fixing $i$, then choosing the $i$ neighbors of 1, and then the forest of size $k - 1 + i$ on $n - 1$ nodes. Therefore,

$$T_{n,k} = \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) T_{n-1,k-1+i}$$

(1)

with the initial conditions $T_{0,0} = 1$ and $T_{n,0} = 0$.

**Proposition.** $T_{n,k} = kn^{n-k-1}$. In particular, the number of labeled trees is $T_{n,1} = T_n = n^{n-2}$.

**Proof.** For a given $n$, suppose this holds for $k - 1 \geq 0$. Then using (1) we have

$$T_{n,k} = \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) T_{n-1,k-1+i}$$

I.H. $\sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) (k - 1 + i)(n - 1)^{n-1-k-i}$

$= \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) (n - 1 - i)(n - 1)^{i-1}$

$= \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) (n - 1)^{i} - \sum_{i=1}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) i(n - 1)^{i-1}$

$= n^{n-k} - (n - k) \sum_{i=1}^{n-k} \left( \begin{array}{c} n-1-k \\ i-1 \end{array} \right) (n - 1)^{i-1}$

$= n^{n-k} - (n - k) \sum_{i=0}^{n-1-k} \left( \begin{array}{c} n-1-k \\ i \end{array} \right) (n - 1)^{i}$

$= n^{n-k} - (n - k)n^{n-1-k}$

$= kn^{n-1-k}$

See [1] for other elegant, and very different proofs of this result.
Euler’s Formula. If $G$ is a connected plane graph with $n$ vertices, $e$ edges and $f$ faces, then

$$n - e + f = 2$$

Proof. Let $T \subset E$ be a subset of edges that forms a spanning tree for $G$. Let $G^*$ denote the dual graph of $G$, with edge set $E^*$.

Consider the set of edges $T^* \subset E^*$ in the dual graph that correspond to edges in $E \setminus T$. Then $T^*$ connects all of the faces, since $T$ does not have a cycle. Also, $T^*$ does not contain a cycle; if it did it would separate some vertices of $G$ inside the cycle from the vertices outside the cycle, which is impossible since $T$ is connected. Thus, $T^*$ is a spanning tree of $G^*$.

Now, the number of vertices in a tree is one larger than the number of edges. Therefore, $n = e_T + 1$. Similarly, $f = e_{T^*} + 1$. Combining these gives

$$n + f = (e_T + 1) + (e_{T^*} + 1) = e + 2.$$
**Corollary.** Suppose that $G$ is a plane graph with $n > 2$ vertices. Then

(a) $G$ has a vertex of degree at most 5;

(b) $G$ has at most $3n - 6$ edges.

**Proof.** Let $f_i$ be the number of faces with $i$ sides, and let $n_j$ be the number of nodes with $j$ neighbors. Then

$$f = f_1 + f_2 + f_3 + f_4 + \cdots$$
$$n = n_0 + n_1 + n_2 + n_3 + \cdots$$

Moreover, since every edge has two endpoints, we have that

$$2e = n_1 + 2n_2 + 3n_3 + 4n_4 + \cdots$$

That is, every edge contributes 2 to the sum of all degrees. Similarly, we have that

$$2e = f_1 + 2f_2 + 3f_3 + 4f_4 + \cdots$$

That is, every edge borders two faces.

Now, since every face must have at least 3 sides, we have that in fact

$$f = f_3 + f_4 + f_5 + \cdots$$
$$2e = 3f_3 + 4f_4 + 5f_5 + \cdots$$

and therefore $2e - 3f \geq 0$.

To prove (a), suppose on the contrary that every vertex had degree at least 6. Then we would have

$$n = n_6 + n_7 + n_8 + \cdots$$
$$2e = 6n_6 + 7n_7 + 8n_8$$

which would imply that $2e - 6n \geq 0$. Combining these gives

$$(2e - 6n) + 2(2e - 3f) = 6(e - f - n) \geq 0$$

which implies $e \geq n + f$, which contradicts Euler’s formula.

To prove (b), we use the relation $2e - 3f \geq 0$ and Euler’s formula to conclude that

$$3n - 6 = 3e - 3f \geq e.$$
Corollary. Every plane graph can be 6-colored.

Proof. By induction on the number of nodes. For small cases with fewer than 6 nodes, this is obvious. Suppose that the statement is true for planar graphs with fewer than \( n \) nodes, and let \( G \) be a plane graph with \( n \) nodes. Then \( G \) has a vertex \( v \) that has degree no larger than 5. Removing \( v \) from \( G \), the resulting graph \( G' = G - \{v\} \) is 6-colorable, by the induction hypothesis. But since \( v \) has no more than 5 neighbors, we can extend the coloring to all of \( G \) by coloring \( v \) different from its neighbors.

References