Floating Point

15-213/18-213/15-513/18-613: Introduction to Computer Systems
4th Lecture, Jan 23, 2020
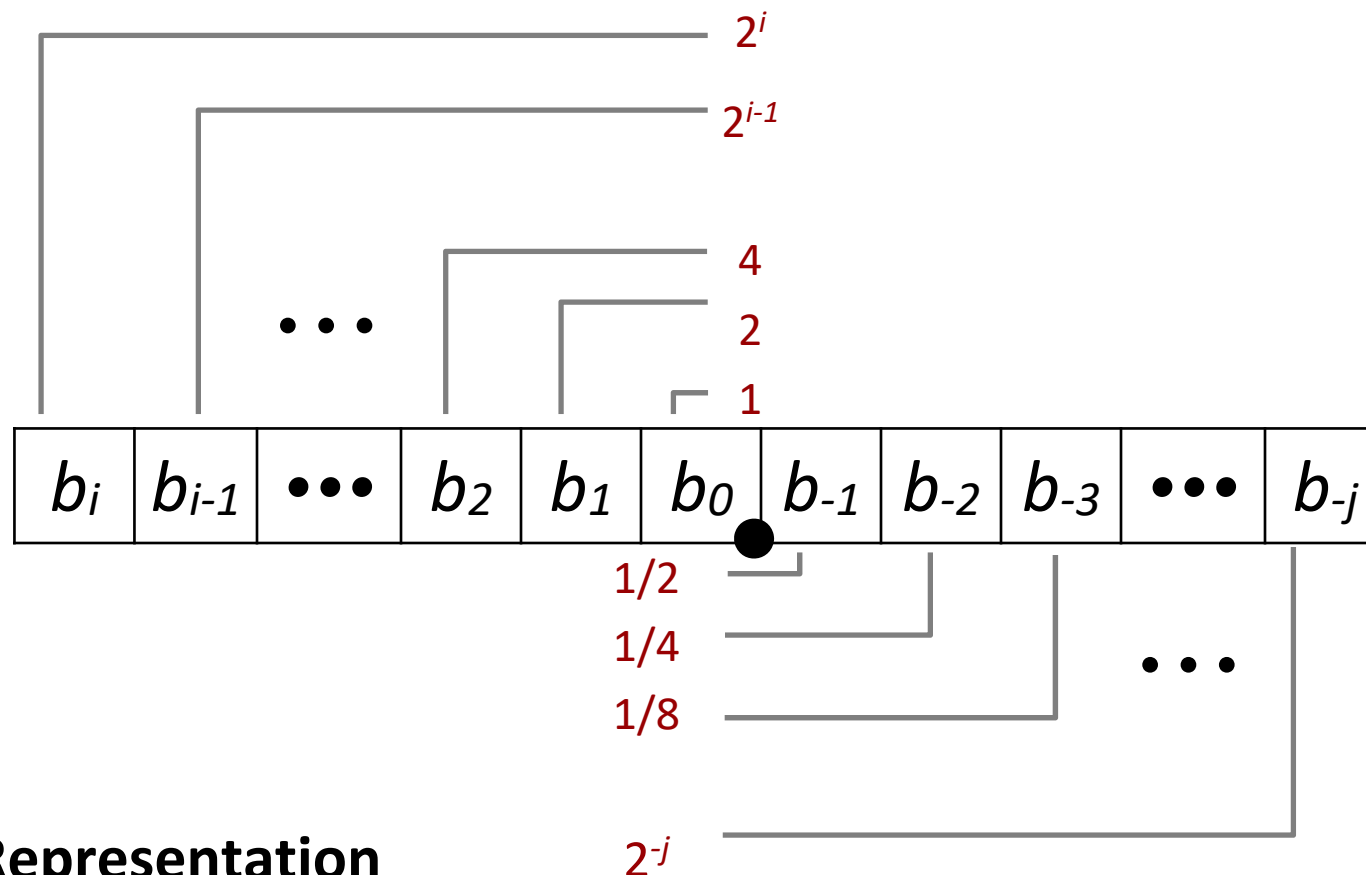
Today: Floating Point

- ■ **Background: Fractional binary numbers**

- ■ **IEEE floating point standard: Definition**

- ■ **Example and properties**

- ■ **Rounding, addition, multiplication**

- ■ **Floating point in C**

- ■ **Summary**

Fractional binary numbers

■ **What is 1011.101_2?**

Fractional Binary Numbers

- 2^i
- 2^{i-1}
- 4
- 2
- 1

| b_i | b_{i-1} | $\bullet\bullet\bullet$ | b_2 | b_1 | b_0 | b_{-1} | b_{-2} | b_{-3} | $\bullet\bullet\bullet$ | b_{-j} |

- $1/2$
- $1/4$
- $1/8$
- 2^{-j}

■ **Representation**

- ▪ Bits to right of "binary point" represent fractional powers of 2
- ▪ Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- **Value**   **Representation**

  5 3/4   = 23/4 101.11_2 = 4 + 1 + 1/2 + 1/4

  2 7/8   = 23/8 10.111_2 = 2 + 1/2 + 1/4 + 1/8

  1 7/16 = 23/16 1.0111_2 = 1 + 1/4 + 1/8 + 1/16

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation #1**
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations

  - Value   Representation
    - 1/3   `0.0101010101[01]`…$_2$
    - 1/5   `0.001100110011[0011]`…$_2$
    - 1/10   `0.0001100110011[0011]`…$_2$

- **Limitation #2**
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

- **IEEE Standard 754**

  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
  - Some CPUs don't implement IEEE 754 in full
   e.g., early GPUs, Cell BE processor

- **Driven by numerical concerns**

  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers
      in defining standard

This is important!

- **Ariane 5 explodes on maiden voyage: $500 MILLION dollars lost**
  - 64-bit floating point number assigned to 16-bit integer
  - Causes rocket to get incorrect value of horizontal velocity and crash
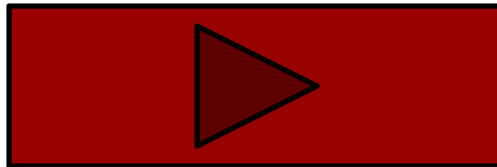
- **Patriot Missile defense system misses scud – 28 people die**
  - System tracks time in tenths of second
  - Converted from integer to floating point number.
  - Accumulated rounding error causes drift. 20% drift over 8 hours.
  - Eventually (on 2/25/1991 system was on for 100 hours) causes range mis-estimation sufficiently large to miss incoming missiles.

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- **Numerical Form:**

$$(-1)^s\ M\ 2^E$$

  - **Sign bit** *s* determines whether number is negative or positive
  - **Significand** *M* normally a fractional value in range [1.0,2.0).
  - **Exponent** *E* weights value by power of two

- **Encoding**

  - MSB s is sign bit *s*
  - **exp** field encodes *E* (but is not equal to E)
  - **frac** field encodes *M* (but is not equal to M)

s	exp	frac

Precision options

■ **Single precision: 32 bits**

≈ 7 decimal digits, $10^{\pm 38}$

s	exp	frac
1	8-bits	23-bits

■ **Double precision: 64 bits**

≈ 16 decimal digits, $10^{\pm 308}$

s	exp	frac
1	11-bits	52-bits

■ **Other formats: half precision, quad precision**

Three "kinds" of floating point numbers

s	exp	frac

1   e-bits   f-bits

00…00

exp ≠ 0 and exp ≠ 11…11

11…11

denormalized   **normalized**   **special**
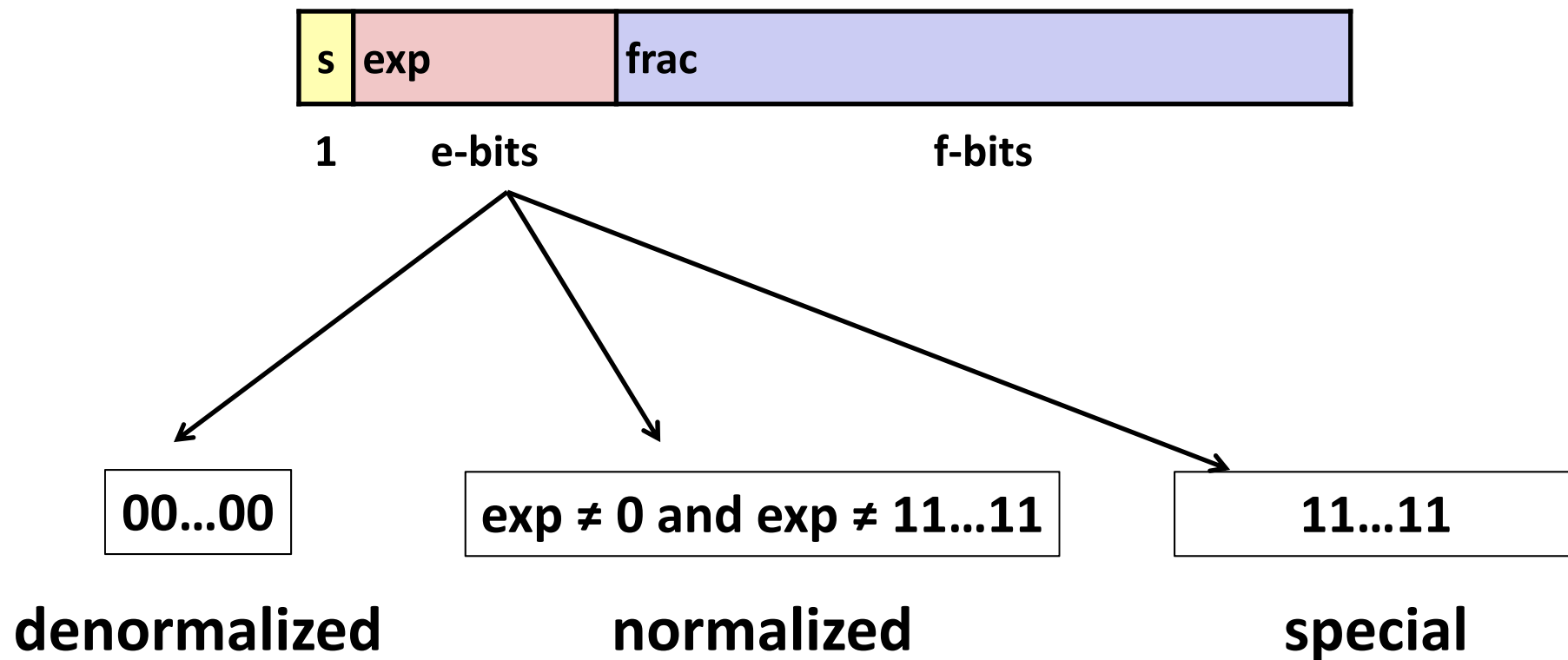
"Normalized" Values

$$v = (-1)^s \, M \, 2^E$$

- **When: `exp` ≠ 000...0 and `exp` ≠ 111...1**

- **Exponent coded as a *biased* value: $E = $ `exp` $-$ *Bias***
  - *exp*: unsigned value of exp field
  - *Bias* = $2^{k-1} - 1$, where k is number of exponent bits
    - Single precision: 127 (`exp`: 1...254, E: -126...127)
    - Double precision: 1023 (`exp`: 1...2046, E: -1022...1023)

- **Significand coded with implied leading 1: $M = 1.xxx...x_2$**
  - xxx...x: bits of frac field
  - Minimum when `frac`=000...0 (M = 1.0)
  - Maximum when `frac`=111...1 (M = 2.0 $-$ ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$$v = (-1)^s M 2^E$$
$$E = exp - Bias$$

- **Value: `float F = 15213.0;`**
  - 15213_{10} = 11101101101101_2

    = $1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = $1.\underline{1101101101101}_2$
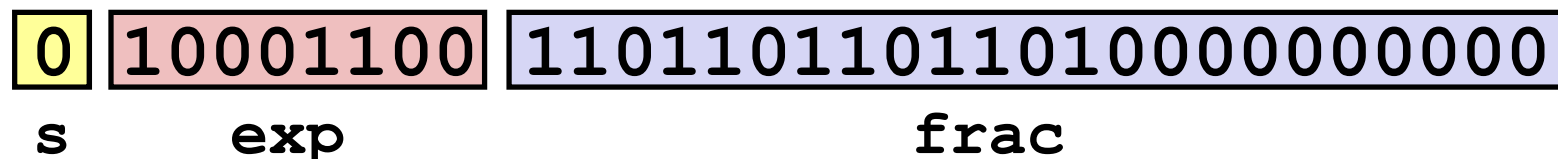
  `frac=` $\underline{1101101101101}0000000000_2$

- **Exponent**

  E = 13

  $Bias$ = 127

  `exp =` 140 = 10001100_2

- **Result:**

0	10001100	11011011011010000000000
s	exp	frac

Denormalized Values

$$v = (-1)^s \, M \, 2^E$$
$$E = 1 - Bias$$

- **Condition:** exp = 000…0

- **Exponent value:** $E = 1 - Bias$ (instead of $\texttt{exp} - Bias$)   (why?)
- **Significand coded with implied leading 0:** $M = 0.xxx…x_2$
  - $\texttt{xxx}…\texttt{x}$: bits of $\texttt{frac}$
- **Cases**
  - $\texttt{exp} = 000…0$, $\texttt{frac} = 000…0$
    - Represents zero value
    - Note distinct values: +0 and −0 (why?)
  - $\texttt{exp} = 000…0$, $\texttt{frac} \neq 000…0$
    - Numbers closest to 0.0
    - Equispaced

Special Values

■ **Condition: `exp` = 111…1**

■ **Case: `exp` = 111…1, `frac` = 000…0**

- **Represents value ∞ (infinity)**
- Operation that overflows
- Both positive and negative
- E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$

■ **Case: `exp` = 111…1, `frac` ≠ 000…0**

- **Not-a-Number (NaN)**
- Represents case when no numeric value can be determined
- E.g., $\text{sqrt}(-1)$, $\infty - \infty$, $\infty \times 0$

C float Decoding Example

$$v = (-1)^s M 2^E$$
$$E = exp - Bias$$

float: `0xC0A00000`

Bias $= 2^{k-1} - 1 = 127$

binary: ___ ___ ___ ___ ___ ___ ___ ___

| 1 | 8-bits | 23-bits |

E =

S =

M =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

$$v = (-1)^s M 2^E =$$

C float Decoding Example #1

$$v = (-1)^s \, M \, 2^E$$
$$E = \texttt{exp} - Bias$$

float: `0xC0A00000`

binary: `1100 0000 1010 0000 0000 0000 0000 0000`

1	1000 0001	010 0000 0000 0000 0000 0000
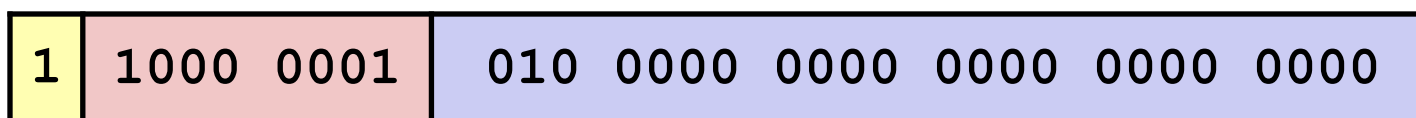
| 1 | 8-bits | 23-bits |

E =

S =

M = 1.

v = (−1)^s M 2^E =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #1

$$v = (-1)^s\, M\, 2^E$$
$$E = \texttt{exp} - Bias$$

float: `0xC0A00000`

$Bias = 2^{k-1} - 1 = 127$

binary: 1100 0000 1010 0000 0000 0000 0000 0000

1	1000 0001	010 0000 0000 0000 0000 0000
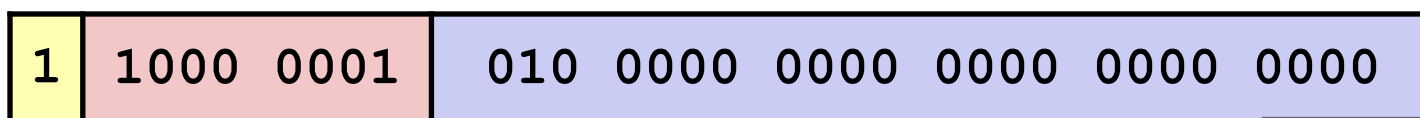
1   8-bits   23-bits

E = exp − Bias = 129 − 127 = 2 (decimal)

S = 1 -> negative number

M = 1.010 0000 0000 0000 0000 0000

   = 1 + 1/4 = 1.25

v = (−1)^s M 2^E = (-1)^1 * 1.25 * 2^2 = -5

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #2

$$v = (-1)^s \, M \, 2^E$$
$$E = 1 - Bias$$

float: `0x001C0000`

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
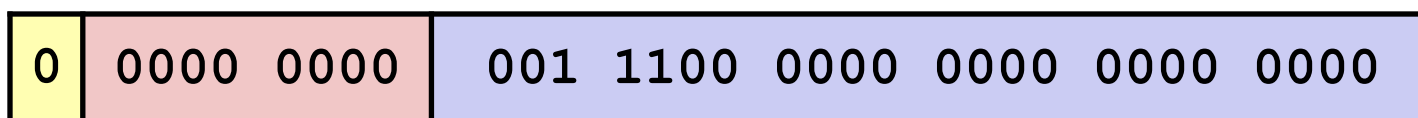
1　　　8-bits　　　　　　　　23-bits

E =

S =

M = 0 .

v = (−1)^s M 2^E =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #2

$$v = (-1)^s\,M\,2^E$$
$$E = 1 - Bias$$

float: `0x001C0000`

$Bias = 2^{k-1} - 1 = 127$

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
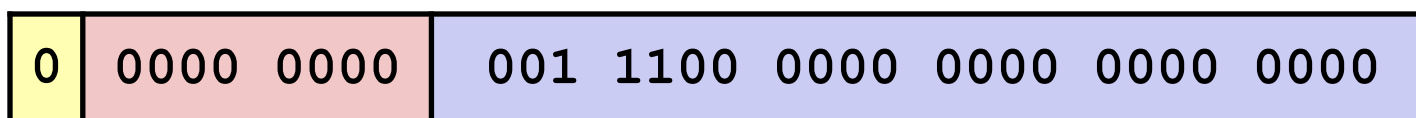
| 1 | 8-bits | 23-bits |

E = 1 − Bias = 1 − 127 = −126 (decimal)

S = 0 -> positive number

M = `0.001 1100 0000 0000 0000 0000`

= 1/8 + 1/16 + 1/32 = 7/32 = 7*2^{-5}

$v = (-1)^s\,M\,2^E = (-1)^0 * 7*2^{-5} * 2^{-126} = 7*2^{-131}$

≈ 2.571393892 X 10^{-39}

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualization: Floating Point Encodings
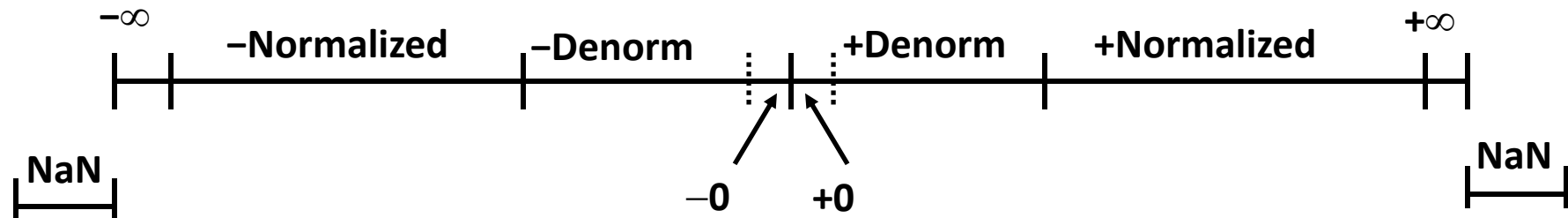
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Tiny Floating Point Example

s	exp	frac

  1   4-bits   3-bits
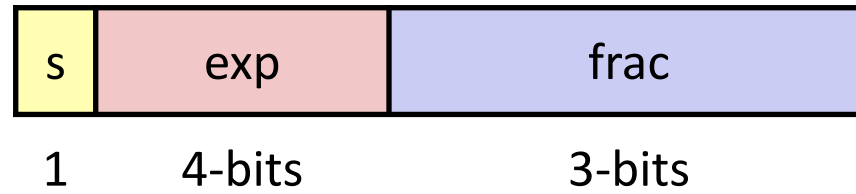
- ## 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the **exp**, with a bias of 7
  - the last three bits are the **frac**

- ## Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (s=0 only)

$$v = (-1)^s M 2^E$$
norm: E = `exp` *– Bias*
denorm: E = 1 – Bias

|	s	exp	frac	E	Value	
	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized	0	0000	010	-6	2/8*1/64 = 2/512	$(-1)^0(0+1/4)*2^{-6}$
numbers	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	$(-1)^0(1+1/8)*2^{-6}$
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized	0	0111	000	0	8/8*1 = 1	
numbers	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- ## 6-bit IEEE-like format
  - e = 3 exponent bits
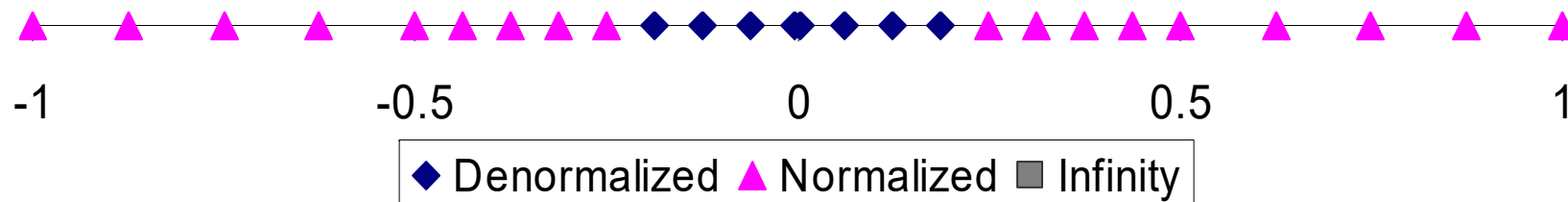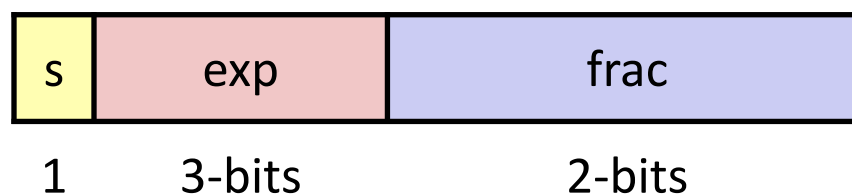 - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

s	exp	frac
1	3-bits	2-bits
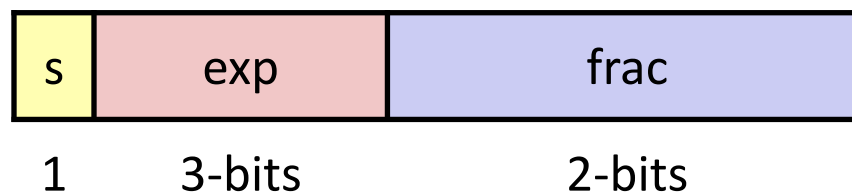
- ## Notice how the distribution gets denser toward zero.

8 values

◆ Denormalized ▲ Normalized ■ Infinity
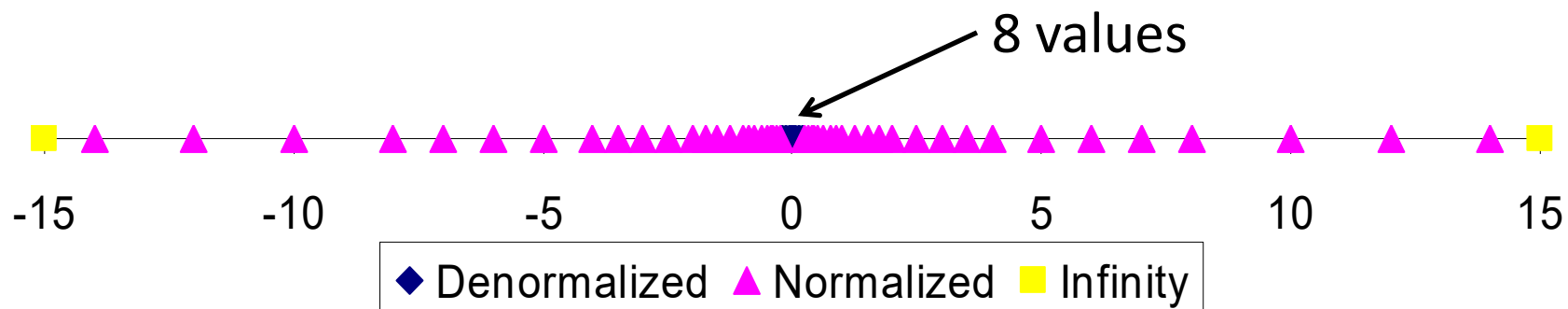
Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3-bits	2-bits

-1   -0.5   0   0.5   1

◆ Denormalized ▲ Normalized ■ Infinity

Special Properties of the IEEE Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield? The answer is complicated.
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Quiz Time!

Check out:

https://canvas.cmu.edu/courses/13182

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■ IEEE floating point standard: Definition

■ Example and properties

■ **Rounding, addition, multiplication**

■ Floating point in C

■ Summary

Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

- $x \times_f y = \text{Round}(x \times y)$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

■ **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**−$1.50**
▪ Towards zero	$1 ↓	$1 ↓	$1 ↓	$2 ↓	−$1 ↑
▪ Round down (−∞)	$1 ↓	$1 ↓	$1 ↓	$2 ↓	−$2 ↓
▪ Round up (+∞)	$2 ↑	$2 ↑	$2 ↑	$3 ↑	−$1 ↑
▪ Nearest Even* (default)	$1 ↓	$2 ↑	$2 ↑	$2 ↓	−$2 ↓

*Round to nearest, but if half-way in-between then round to nearest even

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
    - C99 has support for rounding mode management
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100..._2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Rounding

1.BBG**R**XXX

Guard bit: LSB of result

Sticky bit: OR of remaining bits

Round bit: 1st bit removed

- **Round up conditions**
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Fraction	GRS	Incr?	Rounded
1.0000000	000	N	1.000
1.1010000	100	N	1.101
1.0001000	010	N	1.000
1.0011000	110	Y	1.010
1.0001010	011	Y	1.001
1.1111100	111	Y	10.000

FP Multiplication

- $(-1)^{s1} \, M1 \, 2^{E1} \quad x \quad (-1)^{s2} \, M2 \, 2^{E2}$

- **Exact Result:** $(-1)^{s} \, M \, 2^{E}$
  - Sign s: $s1 \wedge s2$
  - Significand M: $M1 \times M2$
  - Exponent E: $E1 + E2$

- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision

- **Implementation**
  - Biggest chore is multiplying significands

4 bit significand: `1.010*2`^{2} `x 1.110*2`^{3} `= 10.0011*2`^{5}
                        `= 1.00011*2`^{6} `= 1.001*2`^{6}

Floating Point Addition

- **$(-1)^{s1} M1 \; 2^{E1} \; + \; (-1)^{s2} M2 \; 2^{E2}$**
  - Assume $E1 > E2$

- **Exact Result: $(-1)^s M \; 2^E$**
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: $E1$

- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Get binary points lined up

$\longleftarrow E1-E2 \longrightarrow$

$(-1)^{s1} M1$

$+$

$(-1)^{s2} M2$

$(-1)^s M$
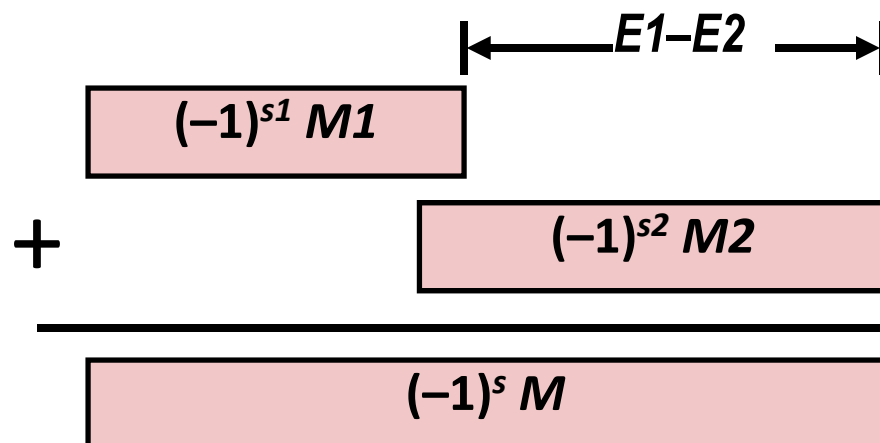
```
1.010*2² + 1.110*2³ = (0.1010 + 1.1100)*2³
= 10.0110 * 2³ = 1.00110 * 2⁴ = 1.010 * 2⁴
```

Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition?   *Yes*
    - But may generate infinity or NaN
  - Commutative?   *Yes*
  - Associative?   *No*
    - Overflow and inexactness of rounding
    - `(3.14+1e10)–1e10 = 0, 3.14+(1e10–1e10) = 3.14`
  - 0 is additive identity?   *Yes*
  - Every element has additive inverse?   *Almost*
    - Yes, except for infinities & NaNs
- **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$?   *Almost*
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication?   *Yes*
    - But may generate infinity or NaN
  - Multiplication Commutative?   *Yes*
  - Multiplication is Associative?   *No*
    - Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)=1e20`
  - 1 is multiplicative identity?   *Yes*
  - Multiplication distributes over addition?   *No*
    - Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 =NaN`
- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$?   *Almost*
    - Except for infinities & NaNs

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Floating Point in C

- ## C Guarantees Two Levels
  - `float` single precision
  - `double` double precision

- ## Conversions/Casting
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
 - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither
d nor **f** is NaN

- `x == (int)(float) x`   ✗
- `x == (int)(double) x`   ✓
- `f == (float)(double) f`   ✓
- `d == (double)(float) d`   ✗
- `f == -(-f);`   ✓
- `2/3 == 2/3.0`   ✗
- `d < 0.0    ⇒    ((d*2) < 0.0)`   ✓
- `d > f      ⇒    -f > -d`   ✓
- `d * d >= 0.0`   ✓
- `(d+f)-d == f`   ✗

Summary

- **IEEE Floating Point has clear mathematical properties**

- **Represents numbers of form M x 2^E**

- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded

- **Not the same as real arithmetic**
  - Violates associativity/distributivity
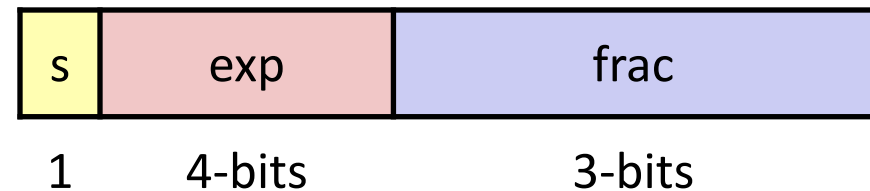 - Makes life difficult for compilers & serious numerical applications programmers

Single precision: 32 bits
Double precision: 64 bits
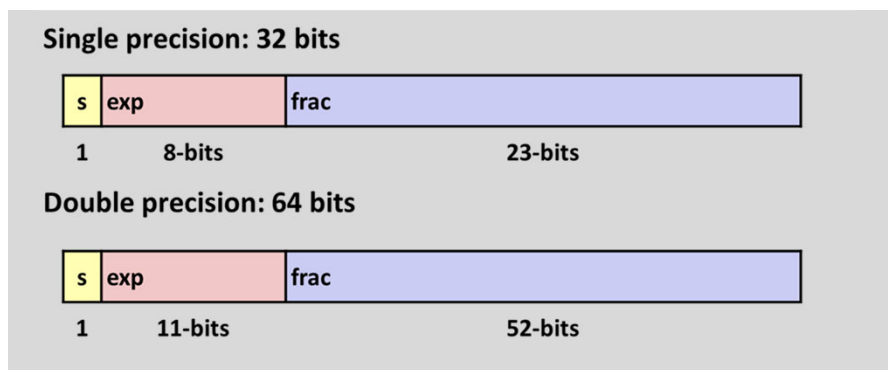
Additional Slides

Creating Floating Point Number

■ **Steps**

■ Normalize to have leading 1

■ Round to fit within fraction

■ Postnormalize to deal with effects of rounding

s	exp	frac
1	4-bits	3-bits

■ **Case Study**

■ Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Normalize

s	exp	frac
1	4-bits	3-bits
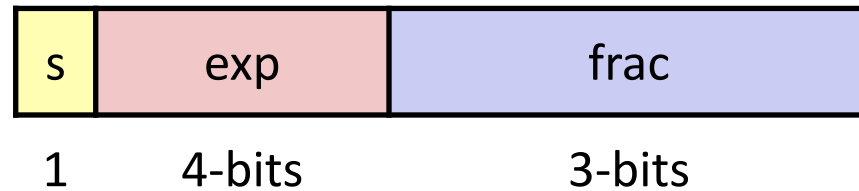
- ## Requirement

  - Set binary point so that numbers of form 1.xxxxx

  - Adjust all to have leading one

    - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Numeric Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

Interesting Numbers {single,double}

Description	exp	frac	Numeric Value
■ **Zero**	00...00	00...00	0.0
■ **Smallest Pos. Denorm.**	00...00	00...01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$

- Single $\approx 1.4 \times 10^{-45}$
- Double $\approx 4.9 \times 10^{-324}$

Description	exp	frac	Numeric Value
■ **Largest Denormalized**	00...00	11...11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$

- Single $\approx 1.18 \times 10^{-38}$
- Double $\approx 2.2 \times 10^{-308}$

Description	exp	frac	Numeric Value
■ **Smallest Pos. Normalized**	00...01	00...00	$1.0 \times 2^{-\{126,1022\}}$

- Just larger than largest denormalized

Description	exp	frac	Numeric Value
■ **One**	01...11	00...00	1.0
■ **Largest Normalized**	11...10	11...11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$

- Single $\approx 3.4 \times 10^{38}$
- Double $\approx 1.8 \times 10^{308}$