Floating Point

15-213: Introduction to Computer Systems
3rd Lecture, Aug. 31, 2010

Instructors:

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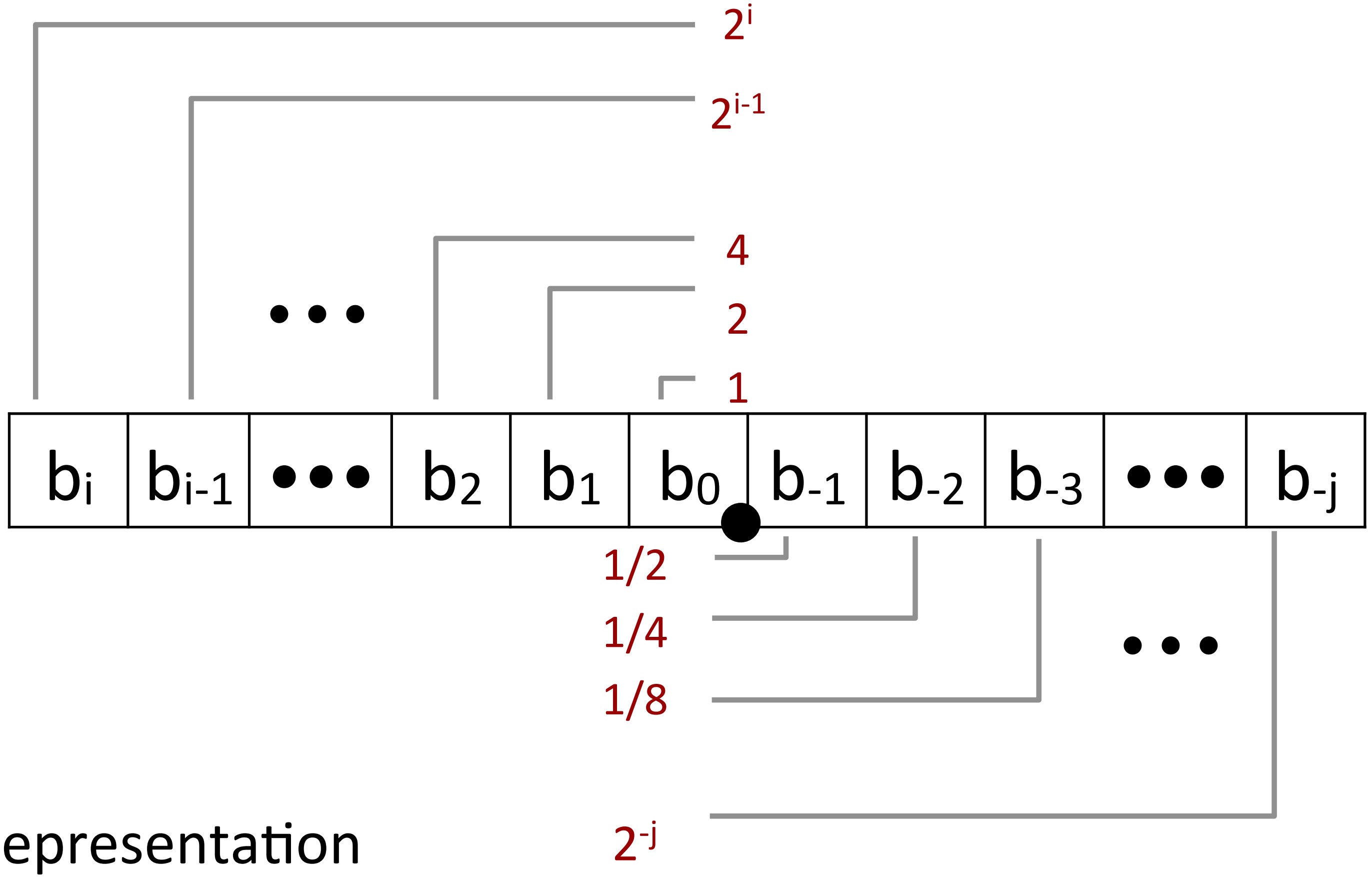
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



2^i

2^{i-1}

4

2

1

b_i b_{i-1} ••• b_2 b_1 b_0 b_{-1} b_{-2} b_{-3} ••• b_{-j}

1/2

1/4

1/8

2^{-j}

- Representation
  - Bits to right of “binary point” represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

Value	Representation	
5 3/4	101.11_2	
2 7/8	10.111_2	
	63/64	1.0111_2

- Observations
- Divide by 2 by shifting right
- Multiply by 2 by shifting left
- Numbers of form $0.111111\ldots_2$ are just below 1.0
  - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
  - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

Value	Representation
1/3	0.0101010101[01]…$_2$
1/5	0.001100110011[0011]…$_2$
1/10	0.0001100110011[0011]…$_2$

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IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- Numerical Form:

$$(-1)^s \, M \;\; 2^E$$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two

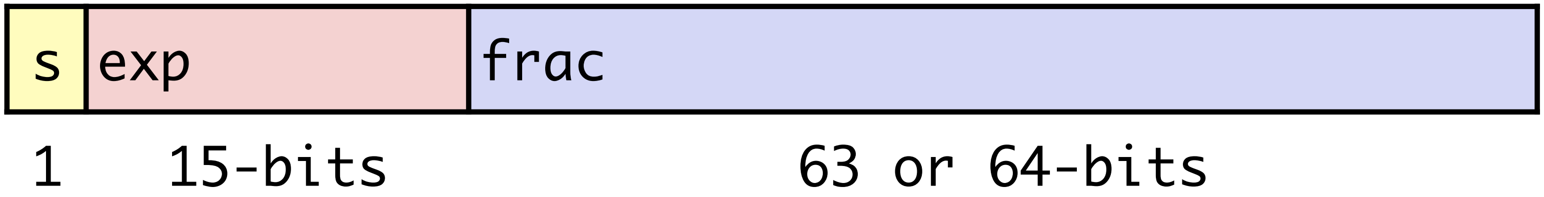
- Encoding
  - MSB `s` is sign bit s
  - `exp` field encodes E (but is not equal to E)
  - `frac` field encodes M (but is not equal to M)

Precisions

- Single precision: 32 bits

- Double precision: 64 bits

- Extended precision: 80 bits (Intel only)



Normalized Values

- Condition: exp ≠ 000…0 and exp ≠ 111…1

- Exponent coded as biased value: E = Exp – Bias
  - Exp: unsigned value `exp`
  - Bias = 2^{k-1} - 1, where k is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of `frac`
  - Minimum when `000…0` (M = 1.0)
  - Maximum when `111…1` (M = 2.0 – ε)
  - Get extra leading bit for “free”

Normalized Encoding Example

- **Value:** `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  $M = 1.\underline{1101101101101}_2$

  $\texttt{frac} = \underline{1101101101101}0000000000_2$

- **Exponent**

  $E = 13$

  $Bias = 127$

  $Exp = 140 = 10001100_2$

- **Result:**

0	10001100	11011011011010000000000
`s`	`exp`	`frac`

Denormalized Values

- Condition: `exp = 000…0`

- Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)
- Significand coded with implied leading 0: $M = 0.xxx…x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = `000…0`, `frac` = `000…0`
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = `000…0`, `frac` ≠ `000…0`
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

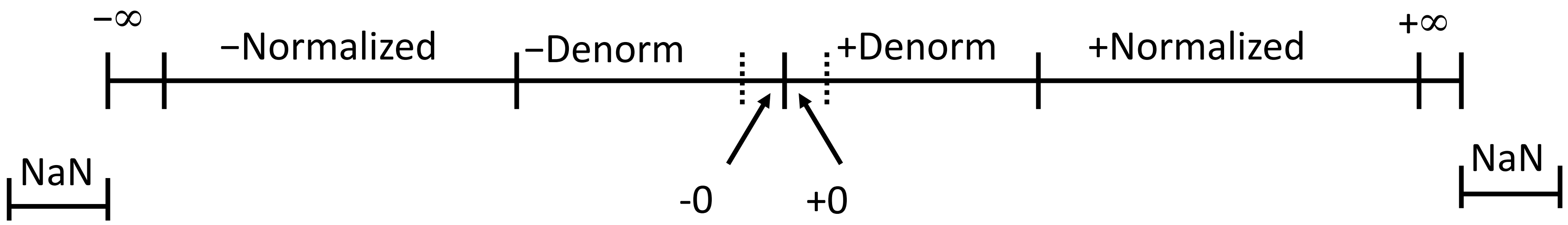
Special Values

- Condition: `exp` = 111…1

- Case: `exp` = 111…1, `frac` = 000…0
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- Case: `exp` = 111…1, `frac` ≠ 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating Point Encodings



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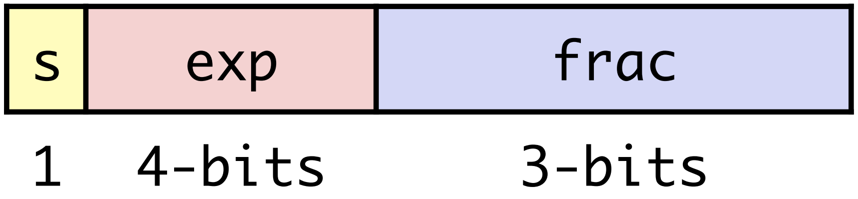
Tiny Floating Point Example



s	exp	frac
1	4-bits	3-bits

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

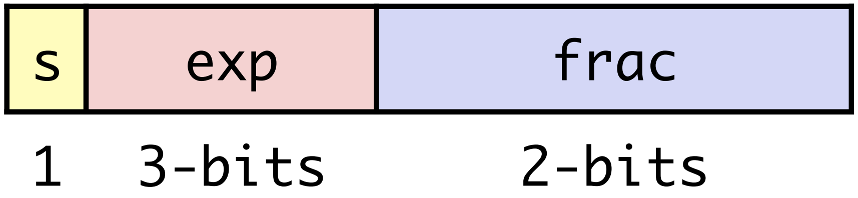
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

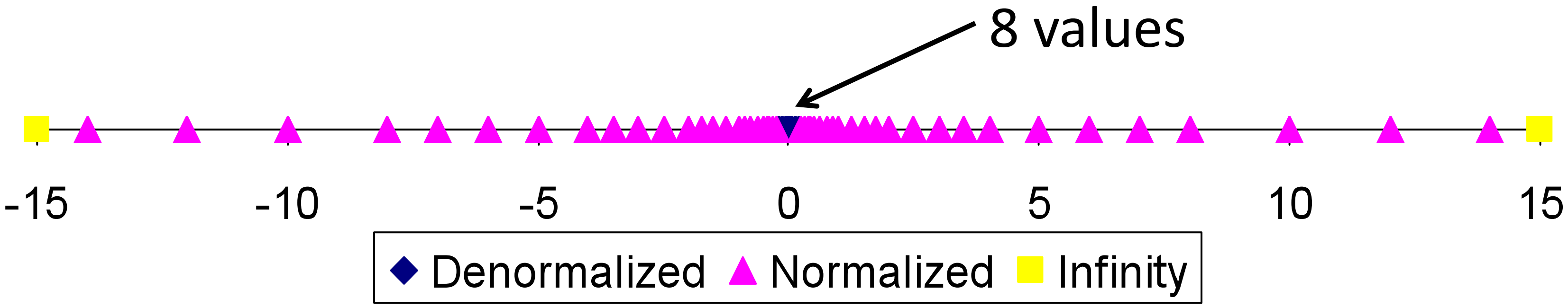
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	**closest to zero**
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	**largest denorm**
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	**smallest norm**
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	**closest to 1 below**
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	**closest to 1 above**
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	**largest norm**
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3

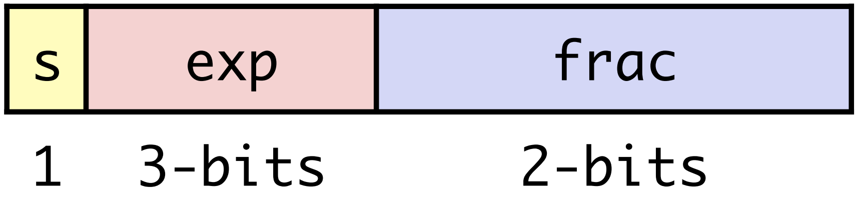


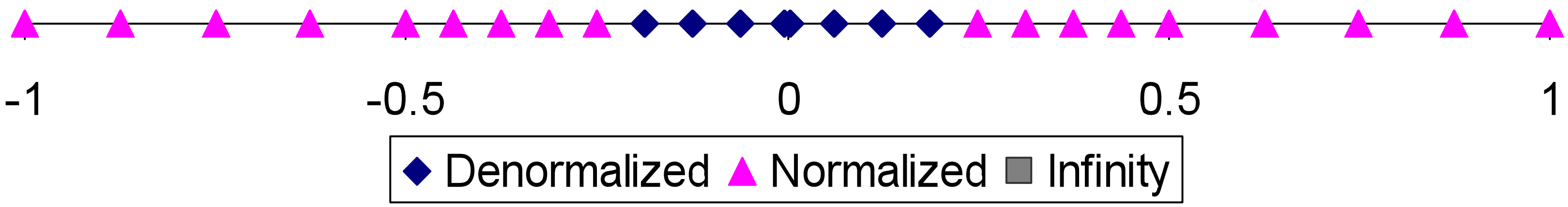
- Notice how the distribution gets denser toward zero.


8 values

Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers `{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Single ≈ 1.4×10^{-45}			
Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Single ≈ 1.18×10^{-38}			
Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
Single ≈ 3.4×10^{38}			
Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- FP Zero Same as Integer Zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- $x +_f y = \mathrm{Round}(x + y)$

- $x \times_f y = \mathrm{Round}(x \times y)$

- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

- Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- What are the advantages of the modes?

Closer Look at Round-To-Even

- Default Rounding Mode
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is `0`
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

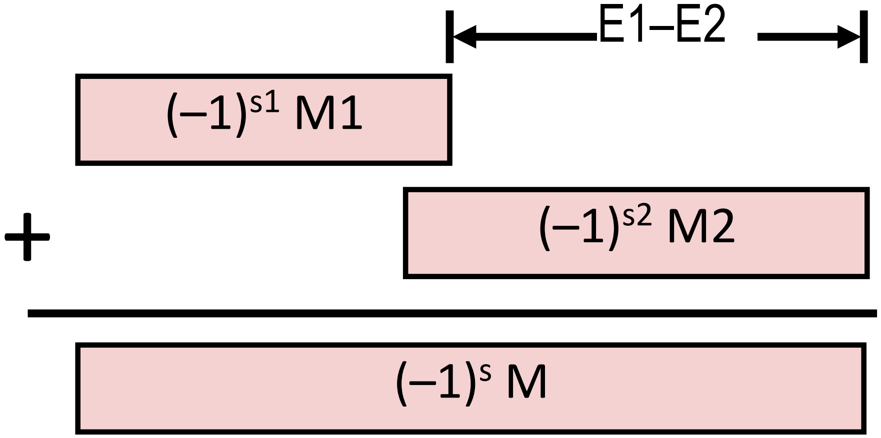
- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \text{x}\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2

- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision

- Implementation
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}$ M1 2^{E1} + $(-1)^{s2}$ M2 2^{E2}
  - Assume E1 > E2
- Exact Result: $(-1)^{s}$ M 2^{E}
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1



E1–E2

$(-1)^{s1}$ M1

+

$(-1)^{s2}$ M2

$(-1)^{s}$ M

- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Mathematical Properties of FP Add

- Compare to those of Abelian Group
  - Closed under addition?
    - But may generate infinity or NaN
  - Commutative?
  - Associative?
    - Overflow and inexactness of rounding
  - 0 is additive identity?
  - Every element has additive inverse
    - Except for infinities & NaNs
- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$?
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- Compare to Commutative Ring
  - Closed under multiplication?
    - But may generate infinity or NaN
  - Multiplication Commutative?
  - Multiplication is Associative?
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity?
  - Multiplication distributes over addition?
    - Possibility of overflow, inexactness of rounding

- Monotonicity
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$?
    - Except for infinities & NaNs

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Floating Point in C

- C Guarantees Two Levels
  - `float` single precision
  - `double` double precision
- Conversions/Casting
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0      ⇒  ((d*2) < 0.0)`
- `d > f        ⇒  -f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

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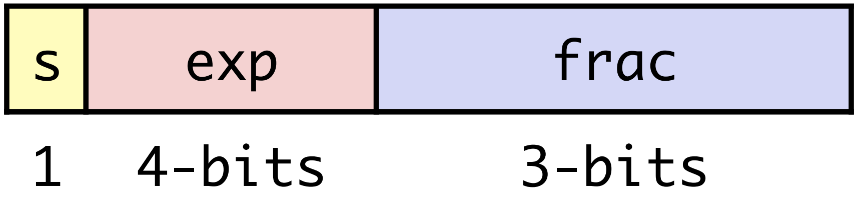
Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
    - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
    - Violates associativity/distributivity
    - Makes life difficult for compilers & serious numerical applications programmers

More Slides

Creating Floating Point Number



s exp frac

1 4-bits 3-bits

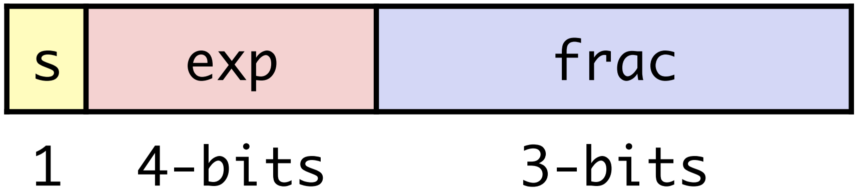
- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- Case Study
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	`10000000`
15	`00001101`
33	`00010001`
35	`00010011`
138	`10001010`
63	`00111111`

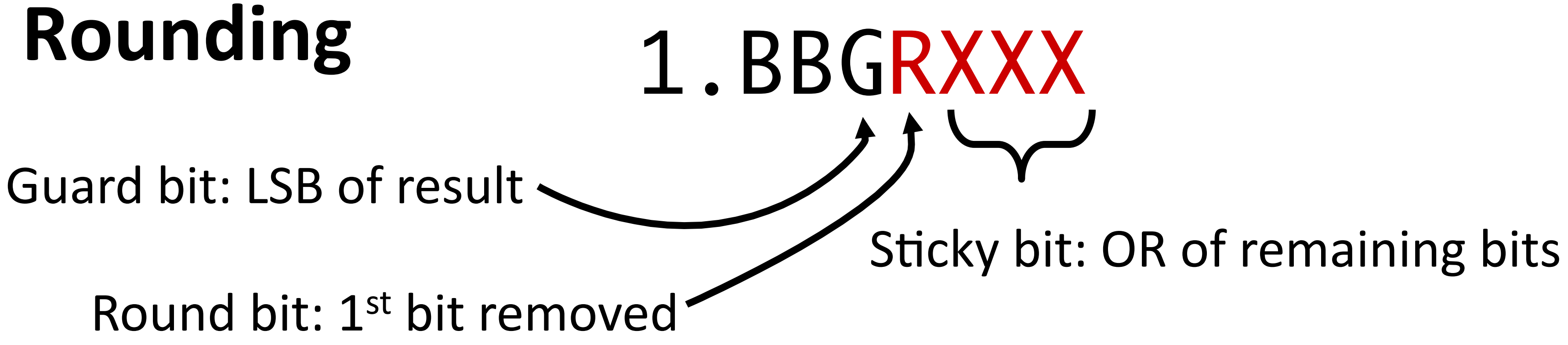
Normalize



- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
`128`	`10000000`	`1.0000000`	`7`
`15`	`00001101`	`1.1010000`	`3`
`17`	`00010001`	`1.0001000`	`4`
`19`	`00010011`	`1.0011000`	`4`
`138`	`10001010`	`1.0001010`	`7`
`63`	`00111111`	`1.1111100`	`5`

Rounding



1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - Round = 1, Sticky = 1 ➙ > 0.5
  - Guard = 1, Round = 1, Sticky = 0 ➙ Round to even

Value	Fraction	GRS	Incr?	Rounded
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64