15-213

"The course that gives CMU its Zip!"

Integer Representations
Sep 5, 2000

Topics

- **Numeric Encodings**
  - Unsigned & Two's complement
- **Programming Implications**
  - C promotion rules

Notation

W: Number of Bits in "Word"

C Data Type	Typical 32-bit	Alpha
`long int`	32	64
`int`	32	32
`short`	16	16
`char`	8	8

Integers

- **Lower case**
- **E.g.,** x, y, z

Bit Vectors

- **Upper Case**
- **E.g.,** X, Y, Z
- **Write individual bits as integers with value 0 or 1**
- **E.g.,** $X = x_{w-1}, x_{w-2}, \ldots x_0$
  - Most significant bit on left

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- C `short` 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

Sign Bit

- For 2's complement, most significant bit indicates sign
  - 0 for nonnegative
  - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Other Encoding Schemes

Other less common encodings

- One's complement: Invert bits for negative numbers
- Sign magnitude: Invert sign bit for negative numbers

- `short int` examples

15213	Unsigned	`00111011 01101101`
-15213	Two's complement	`11000100 10010011`
-15213	One's complement	`11000100 10010010`
-15213	Sign magnitude	`10111011 01101101`

- ISO C does not define what encoding machines use for signed integers, but 99% (or more) use two's complement.
- For truly portable code, don't count on it.

Numeric Ranges

Unsigned Values

- *UMin* = 0

  000…0

- *UMax* = $2^w - 1$

  111…1

Two's Complement Values

- *TMin* = -2^{w-1}

  100…0

- *TMax* = $2^{w-1} - 1$

  011…1

Other Values

- Minus 1

  111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	**255**	**65,535**	**4,294,967,295**	**18,446,744,073,709,551,615**
TMax	**127**	**32,767**	**2,147,483,647**	**9,223,372,036,854,775,807**
TMin	**-128**	**-32,768**	**-2,147,483,648**	**-9,223,372,036,854,775,808**

Observations

- |*TMin* | = *TMax* + 1
  - **Asymmetric range**
- *UMax* = 2 * *TMax* + 1

C Programming

- **`#include <limits.h>`**
  - K&R, App. B11
- **Declares constants, e.g.,**
  - `ULONG_MAX`
  - `LONG_MAX`
  - `LONG_MIN`
- **Values platform-specific**

Unsigned & Signed Numeric Values

X	B2U(X)	B2T(X)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

Example Values

- $W = 4$

Equivalence

- **Same encodings for nonnegative values**

Uniqueness

- **Every bit pattern represents unique integer value**
- **Each representable integer has unique bit encoding**

$\Rightarrow$ Can Invert Mappings

- **U2B(x) = B2U^{-1}(x)**
  - Bit pattern for unsigned integer
- **T2B(x) = B2T^{-1}(x)**
  - Bit pattern for two's comp integer

Casting Signed to Unsigned

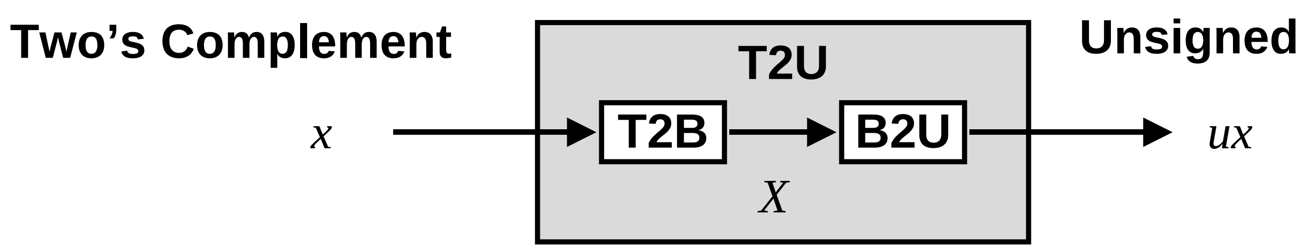
C Allows Conversions from Signed to Unsigned

```
short int           x =  15213;
unsigned short int ux = (unsigned short) x;
short int           y  = -15213;
unsigned short int uy = (unsigned short) y;
```

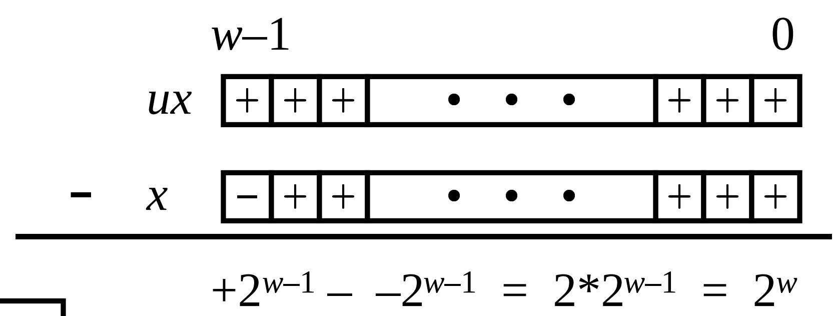
Resulting Value

- **No change in bit representation**
- **Nonnegative values unchanged**
  - *ux* = 15213
- **Negative values change into (large) positive values**
  - *uy* = 50323

Relation Between 2's Comp. & Unsigned



Maintain Same Bit Pattern



$$ux = \begin{cases} x & x \geq 0 \\ x + 2^w & x < 0 \end{cases}$$

Relation Between Signed & Unsigned

Weight	-15213		50323	
1	1	1	1	1
2	1	2	1	2
4	0	0	0	0
8	0	0	0	0
16	1	16	1	16
32	0	0	0	0
64	0	0	0	0
128	1	128	1	128
256	0	0	0	0
512	0	0	0	0
1024	1	1024	1	1024
2048	0	0	0	0
4096	0	0	0	0
8192	0	0	0	0
16384	1	16384	1	16384
32768	1	*-32768*	1	*32768*
Sum		**-15213**		**50323**

- *uy* = *y* + 2 * 32768 = *y* + 65536

From Two's Complement to Unsigned

- **T2U(x)**
  = B2U(T2B(x))
  = $x + x_{w-1} 2^w$

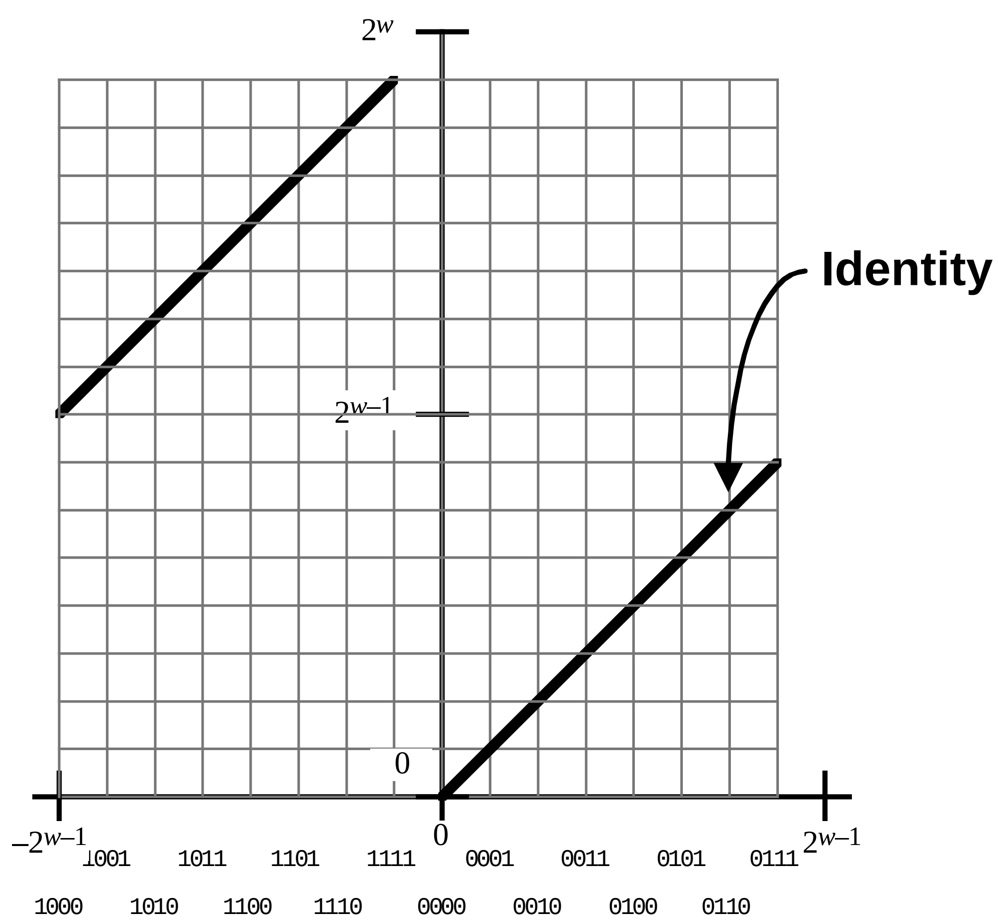
- **What you get in C:**

```
unsigned t2u(int x)
{
  return (unsigned) x;
}
```

X	B2U(X)	B2T(X)
`1010`	10	–6

+ 16

T2U(x)



From Unsigned to Two's Complement

- **U2T(*x*)**

  = B2T(U2B(*x*))

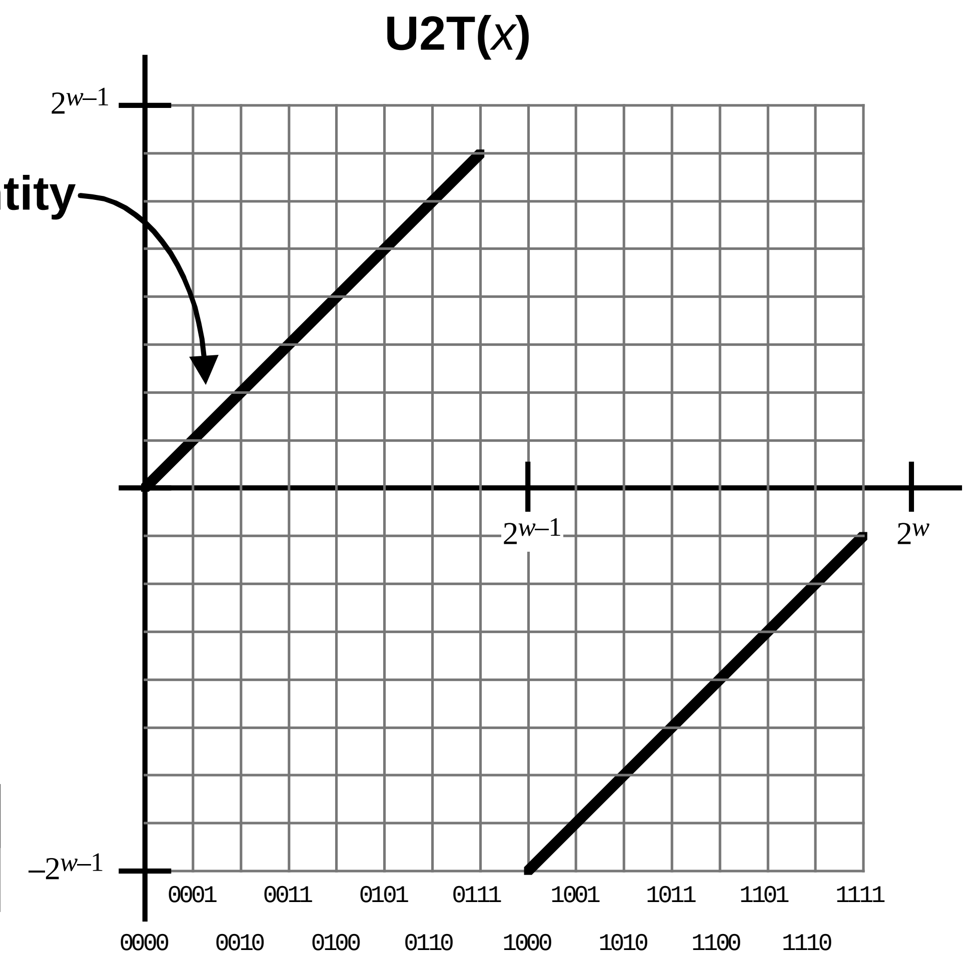
  = $x - x_{w-1} 2^w$

- **What you get in C:**

```
int u2t(unsigned x)
{
  return (int) x;
}
```

X	B2U(*X*)	B2T(*X*)
1010	10	–6

— – 16 →



Signed vs. Unsigned in C

Constants

- **By default are considered to be signed integers**
- **Unsigned if have "U" as suffix**

```
0U, 4294967259U
```

Casting

- **Explicit casting between signed & unsigned same as U2T and T2U**

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- **Implicit casting also occurs via assignments and procedure calls**

```
tx = ux;
uy = ty;
```

Casting Surprises

Expression Evaluation

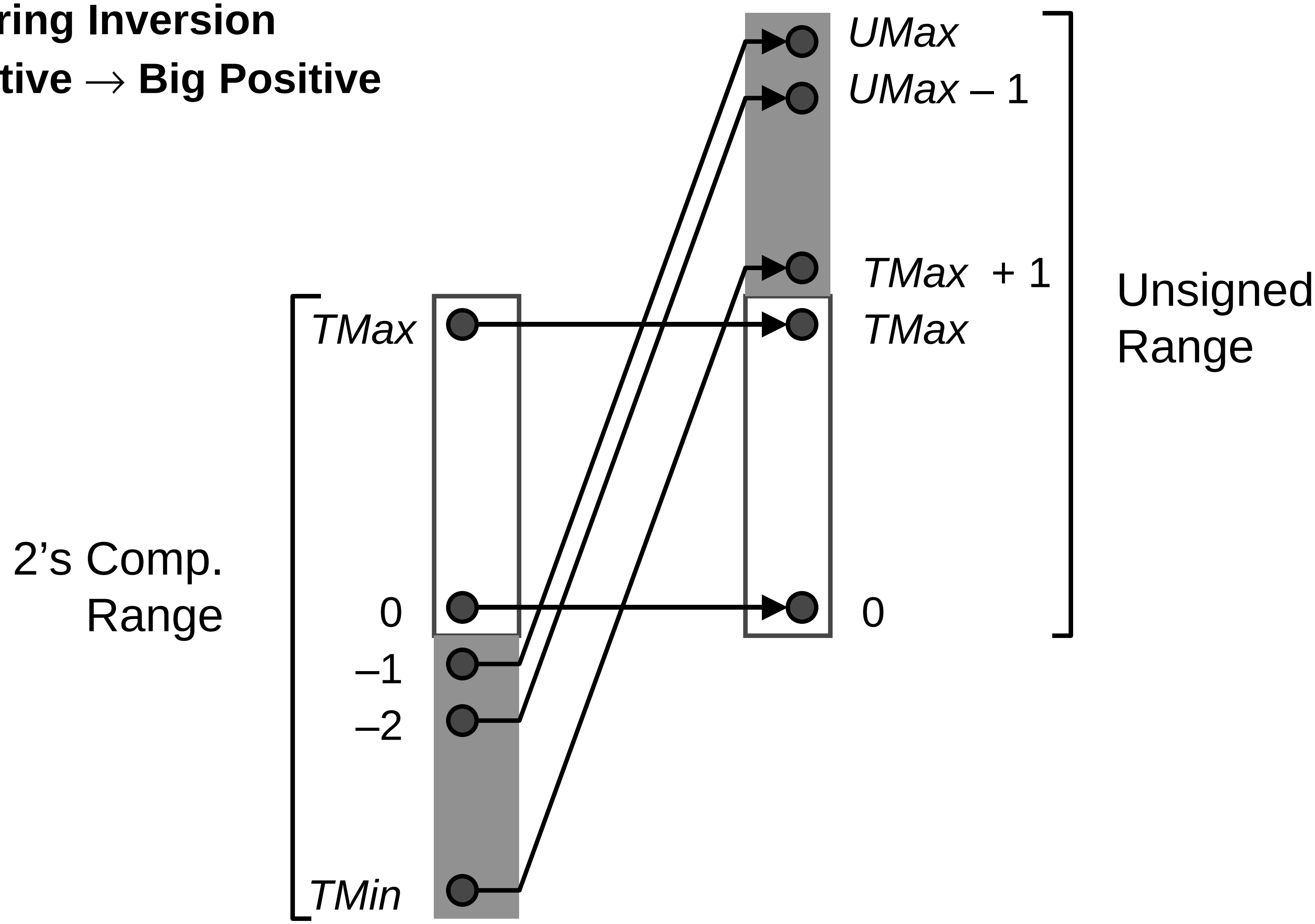
- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples for *W* = 32

Constant_1	Constant_2	Relation	Evaluation
`0`	`0U`	`==`	unsigned
`-1`	`0`	`<`	signed
`-1`	`0U`	`>`	unsigned
`2147483647`	`-2147483648`	`>`	signed
`2147483647U`	`-2147483648`	`<`	unsigned
`-1`	`-2`	`>`	signed
`(unsigned) -1`	`-2`	`>`	unsigned
`2147483647`	`2147483648U`	`<`	unsigned
`2147483647`	`(int) 2147483648U`	`>`	signed

Explanation of Casting Surprises

2's Comp. $\rightarrow$ Unsigned

- **Ordering Inversion**
- **Negative $\rightarrow$ Big Positive**



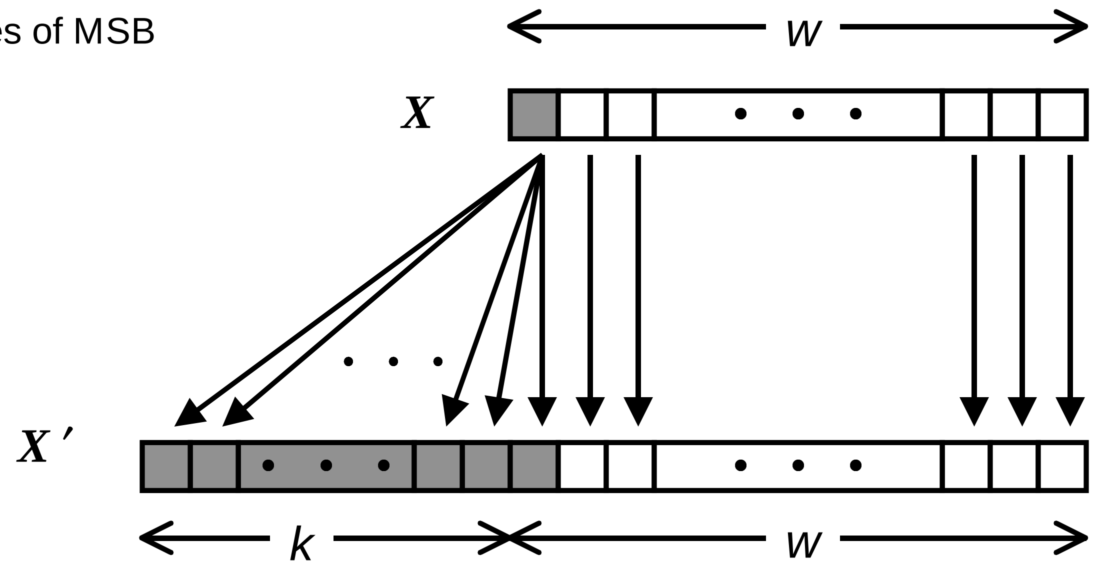
Sign Extension

Task:

- **Given *w*-bit signed integer *x***
- **Convert it to *w*+*k*-bit integer with same value**

Rule:

- **Make *k* copies of sign bit:**
- $X' = \underbrace{x_{w-1},\ldots,x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2},\ldots,x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

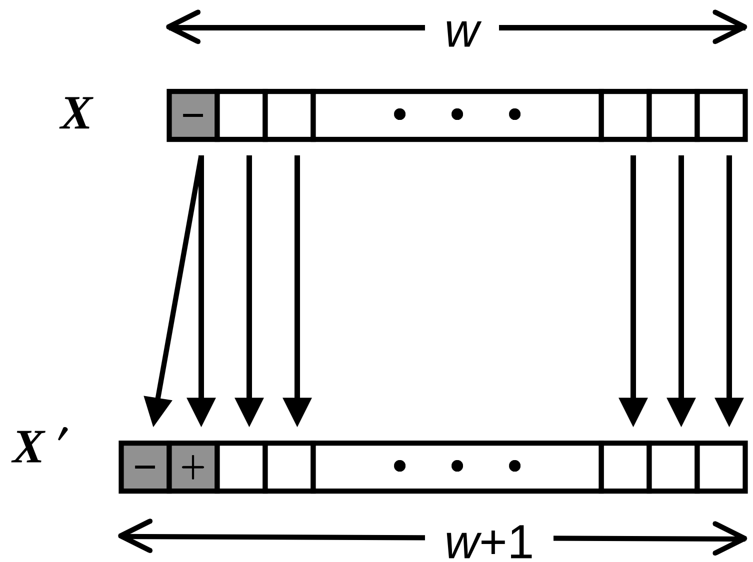
	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 C4 92	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension**

Justification For Sign Extension

Prove Correctness by Induction on *k*

- **Induction Step: extending by single bit maintains value**



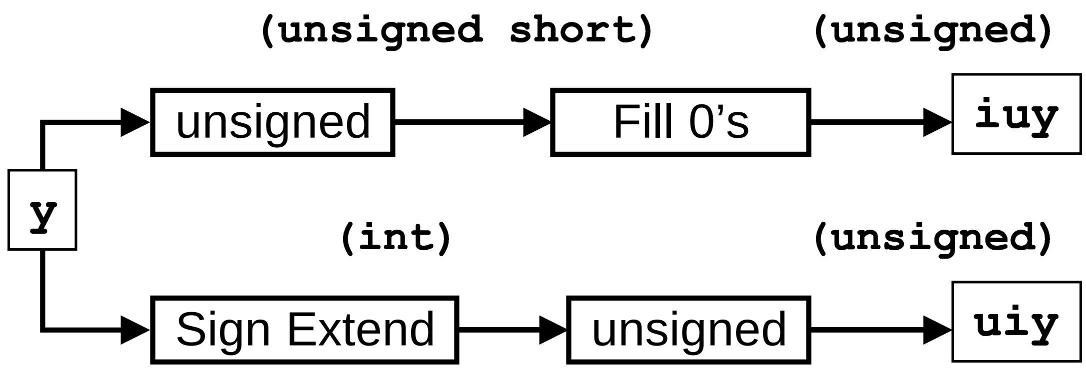
- **Key observation:** $-2^{w-1} = -2^{w} + 2^{w-1}$
- **Look at weight of upper bits:**

X $\quad -2^{w-1} x_{w-1}$

X' $\quad -2^{w} x_{w-1} + 2^{w-1} x_{w-1} \quad = \quad -2^{w-1} x_{w-1}$

Casting Order Dependencies

```
short int x  =  15213;
short int y  = -15213;
unsigned iux = (unsigned)(unsigned short) x;
unsigned iuy = (unsigned)(unsigned short) y;
unsigned uix = (unsigned) (int) x;
unsigned uiy = (unsigned) (int) y;
unsigned uuy = y;
```




```
iux =      15213: 00000000 00000000 00111011 01101101
iuy =      50323: 00000000 00000000 11000100 10010011
uix =      15213: 00000000 00000000 00111011 01101101
uiy = 4294952083: 11111111 11111111 11000100 10010011
uuy = 4294952083: 11111111 11111111 11000100 10010011
```

Why Should I Use Unsigned?

Don't Use Just Because Number Nonzero

- **C compiler on Alpha generates less efficient code**
  - Comparable code on Intel/Linux

```
unsigned i;
for (i = 1; i < cnt; i++)
  a[i] += a[i-1];
```

- **Easy to make mistakes**

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

Do Use When Performing Modular Arithmetic

- **Multiprecision arithmetic**
- **Other esoteric stuff**

Do Use When Need Extra Bit's Worth of Range

- **Working right up to limit of word size**

Negating with Complement & Increment

In C

`~x + 1 == -x`

Complement

- **Observation:** `~x + x == 1111…11`$_2$ `== -1`

x	1	0	0	1	1	1	0	1
+ ~x	0	1	1	0	0	0	1	0
-1	1	1	1	1	1	1	1	1

Increment

- $\sim x + \cancel{x} + (-\cancel{x} + 1) \quad == \quad \cancel{-1} + (-x + \cancel{1})$
- `~x + 1` `==` `-x`

Warning: Be cautious treating `int`'s as integers

- **OK here**

Comp. & Incr. Examples

x = 15213

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`~x`	-15214	`C4 92`	`11000100 10010010`
`~x+1`	-15213	`C4 93`	`11000100 1001001`**`1`**
`y`	-15213	`C4 93`	`11000100 10010011`

0

	Decimal	Hex	Binary
`0`	0	`00 00`	`00000000 00000000`
`~0`	-1	`FF FF`	`11111111 11111111`
`~0+1`	0	`00 00`	`00000000 00000000`