Recitation 6

Treaps

6.1 Announcements

• Midterm 1 is on Friday. You are allowed a single, double-sided, 8.5 × 11 in sheet of paper for notes.

• FingerLab will be released Friday afternoon.
6.2 Example

Recall that a treap is a BST with a priority function $p : U \rightarrow \mathbb{Z}$, where $U$ is the universe of keys. You should think of $p$ as a random number generator: for each key, it returns a random integer. A treap has two structural properties:

1. **BST invariant**: For every $\text{Node}(L, k, R)$, we have $\ell < k$ for every $\ell$ in $L$, and symmetrically $k < r$ for every $r$ in $R$.

2. **Heap invariant**: For every $\text{Node}(L, k, R)$, we have that $p(k) > p(x)$ for every $x$ in either $L$ or $R$.

**Task 6.1.** Build a treap from the following keys and priorities using two different strategies, and observe that the resulting treap is the same in both cases.

1. Run quicksort, creating a new node every time a pivot is chosen.

2. Beginning with an empty tree, sequentially insert keys in priority-order. Each newly inserted key should be placed at a leaf.

<table>
<thead>
<tr>
<th>$k$</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(k)$</td>
<td>5</td>
<td>7</td>
<td>3</td>
<td>2</td>
<td>8</td>
<td>4</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
6.3 Deletion

Consider the following strategy for deleting a key $k$ from a treap:

1. Locate the node containing $k$,
2. Set the priority of $k$ to be $-\infty$ (note that if $k$ has children, then this breaks the heap invariant of the treap),
3. Restore the heap invariant by rotating $k$ downwards until it has only leaves for children,
4. Delete $k$ by replacing its node with a leaf.

A “rotation” in this case refers to the process of making one of $k$’s children the root, depending on their relative priorities. For example, if $k$ has two children with priorities $p_1$ and $p_2$ where $p_1 > p_2$, we rotate like so:

![Diagram of rotation]

The case of $p_1 < p_2$ is symmetric. In turns out that this process is equivalent to calling join on the children of $k$. You should convince yourself of this.

We’re interested in the following: in expectation, how many rotations must we perform before we can delete $k$?
Let’s set up the specifics: we have a treap $T$ formed from the sorted sequence of keys $S$, $|S| = n$. We’re interested in deleting the key $S[d]$. Let $T'$ be the same treap, except that the priority of $S[d]$ is now $-\infty$.

We need a couple indicator random variables:

\[
A^i_j = \begin{cases} 
1, & \text{if } S[i] \text{ is an ancestor of } S[j] \text{ in } T \\
0, & \text{otherwise}
\end{cases}
\]

\[
(A')^i_j = \begin{cases} 
1, & \text{if } S[i] \text{ is an ancestor of } S[j] \text{ in } T' \\
0, & \text{otherwise}
\end{cases}
\]

**Task 6.2.** Write $R_d$, the number of rotations necessary to delete $S[d]$, in terms of the given random variables.

**Task 6.3.** Give $E\left[A^i_d\right]$ and $E\left[(A')^i_d\right]$ in terms of $i$ and $d$.

**Task 6.4.** Compute $E\left[R_d\right]$. For simplicity, you may assume $1 \leq d \leq n - 2$. 

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6.4 Additional Exercises

**Exercise 6.5.** Describe an algorithm for inserting an element into a treap by “undoing” the deletion process described in Section 6.3.

**Exercise 6.6.** For treaps, suppose you are given implementations of find, insert, and delete. Implement split and joinMid in terms of these functions. You’ll need to “hack” the keys and priorities; i.e., assume you can do funky things like insert a key with a specific priority.

**Exercise 6.7.** Given a set of key-priority pairs \((k_i, p_i) : 0 \leq i < n\) where all of the \(k_i\)'s are distinct and all of the \(p_i\)'s are distinct, prove that there is a unique corresponding treap \(T\).