Recitation 4

Scan Reloaded

4.1 Announcements

- *BignumLab* has been released, and is due **Friday afternoon**. It's worth 175 points.
- *RandomLab* will be released on Friday.

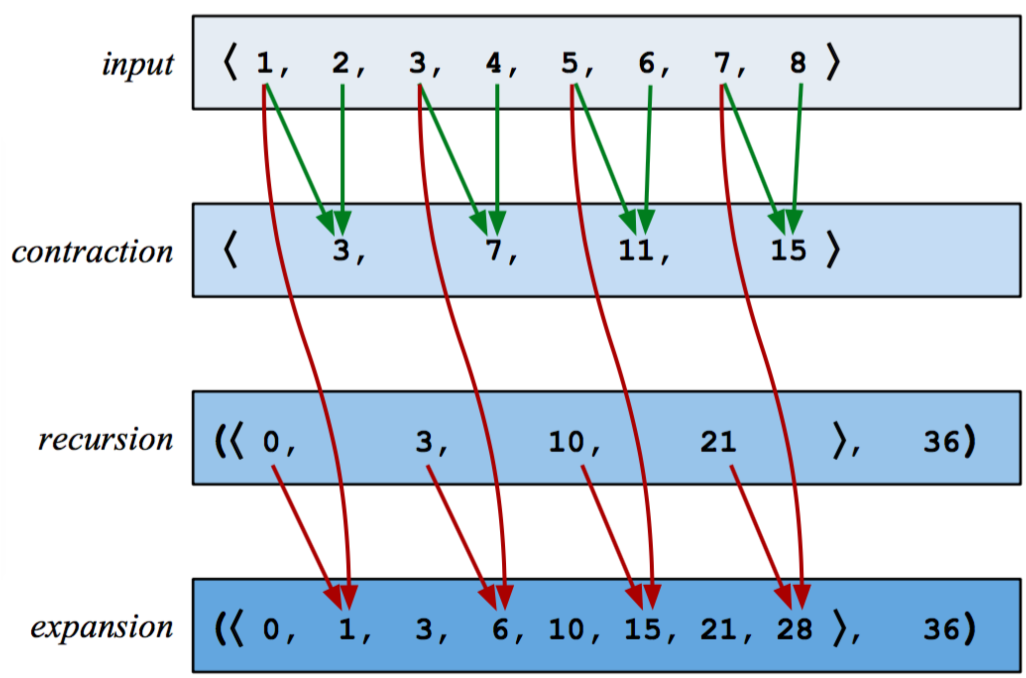
4.2 Implementation

Recall the implementation of `scan` for sequences of power-of-2 length. Note that we typically refer to line 7 as the *contraction* step, line 8 as the *recursive* step, and line 11 as the *expansion* step.

Algorithm 4.1. *scan, assuming $|S|$ is a power of 2.*

```
1  fun scan f b S =
2    case |S| of
3      0 ⇒ (⟨ ⟩, b)
4    | 1 ⇒ (⟨b⟩, S[0])
5    | n ⇒
6        let
7          val S' = ⟨f(S[2i], S[2i+1]) : 0 ≤ i < n/2⟩
8          val (R, t) = scan f b S'
9          fun P(i) = if even(i) then R[i/2] else f(R[⌊i/2⌋], S[i−1])
10       in
11         (⟨P(i) : 0 ≤ i < n⟩, t)
12       end
```

A diagram should help clear up any confusion. Consider $(\texttt{scan} + 0 \ \langle 1,2,3,4,5,6,7,8\rangle)$.



4.3 Cost Analysis

Since we so commonly use `scan` with a constant-time function argument, it is helpful to memorize that it has $O(n)$ work and $O(\log n)$ span in this case. But what about more complex functions? Let's try `merge` as an example.

Task 4.2. *Analyze the work and span of*

$$\mathtt{scan}\ (\mathtt{merge}\ \mathtt{cmp})\ \langle\,\rangle\ S$$

assuming that $|S| = n$, $|x| \leq m$ *for every* $x \in S$, *and* $\mathtt{cmp}$ *is constant-time. Give your answers as tight Big-O bounds in terms of* n *and* m.

Recall that $(\mathtt{merge\ cmp}\ (A, B))$ requires $O(|A| + |B|)$ work and $O(\log |A| + \log |B|)$ span, and it produces a sequence of length $|A| + |B|$.

4.4 Bonus Exercise: Factorials with Bignums

In this section, we write $**$ for bignum multiplication and $\overline{x}$ for the bignum representation of x. We'll be using the same conventions here as in *BignumLab*.

Factorials quickly become too large to represent in a single 32-bit or 64-bit unsigned integer.[1] This makes them the perfect candidate for bignums, which can be arbitrarily large. Consider the following code, which computes the first n factorials (excluding $0!$):

Algorithm 4.3. *Bignum Factorials.*

fun $\mathtt{factorials}\ n = \mathtt{Seq.scanIncl}\ **\ \overline{1}\ \langle \overline{i} : 1 \leq i \leq n \rangle$

Exercise 4.4. *Analyze the work of (*`factorials n`*). Note that you'll first need to determine*

1. *The work of* $\overline{x} ** \overline{y}$*, and*
2. *The bit width of* $\overline{x} ** \overline{y}$*.*

The former is given by solving the recurrence given in BignumLab *for multiplication, namely*

$$W(n) = 3\,W\left(\frac{n}{2}\right) + O(n).$$

The latter can be determined via a little bit of algebra. Note that the bit width of a number $\overline{x}$ *is* $1 + \lfloor \log_2 x \rfloor$*, assuming* $x \geq 1$*.*

Warning: this is pretty hard.

[1] With 32-bit unsigned integers, the largest factorial we can compute before encountering overflow is $11!$. For 64-bits, it's $19!$.