Recitation 5 – *Probability*

Parallel and Sequential Data Structures and Algorithms, 15-210 (Spring 2015)

February 10th, 2015

1 Announcements

- How did *Bignum* go?
- *RandomLab* is out! This lab has a greater written component than previous ones, so we will go over some probability concepts that we think will be useful.
- Questions about homework or lecture?

2 Distributions

We will begin by introducing and/or reminding you of some common probability distributions. For each we will derive the probability described by that distribution and the expected value of the random variable for that distribution. As a reminder, the definition for the expected value of a random variable is $\mathbb{E}(X) = \sum_{x \in X} x\mathbf{Pr}(X = x)$.

2.1 Bernoulli

The Bernoulli distribution is a probability distribution of a random variable X that takes on two values: 1 with (success) probability p and 0 with (failure) probability $1 - p$.

- $\mathbf{Pr}(X = 1) =$
- $\mathbb{E}(X) =$

2.2 Binomial

The binomial distribution serves to find the probability of k successes happening in n Bernoulli trials, each with probability p of succeeding.

- $\mathbf{Pr}(X = k) =$

  Note: there are two approaches to this derivation; one might be easier than the other.

- $\mathbb{E}(X) =$

2.3 Geometric

The geometric distribution describes the distribution of a random variable X that takes on the number of Bernoulli trials k required to get a success (where each Bernoulli trial has a probability p of succeeding).

- $\mathbf{Pr}(X = k) =$

- $\mathbb{E}(X) =$

3 Total Expectation Theorem

We introduce the Total Expectation Theorem that, among other uses, allows us to calculate the expected value of a geometric R.V. much less painfully. First, we need to talk about the conditional expectation of a random variable. Given R.V.'s X and Y, the conditional expectation is defined as

$\mathbb{E}(X|Y = y) = \sum_{x \in X} x\mathbf{Pr}(X = x|Y = y)$

The Total Expectation Theorem uses this result to define the expected value of an R.V. X as

$\mathbb{E}(X) = \sum_{y \in Y} \mathbf{Pr}(Y = y)\mathbb{E}(X|Y = y)$

provided that the elements of an R.V. Y, $y \in Y$ partition the space and X, Y are drawn from the same probability space.

3.1 Geometric, take 2

Now let's use the TET to find the expected value of a geometric random variable X.

$$\begin{aligned}\mathbb{E}(X) &= \mathbf{Pr}(X=1)\mathbb{E}(X|X=1) + \mathbf{Pr}(X>1)\mathbb{E}(X|X>1) \\ &= \end{aligned}$$

4 Applications

4.1 Quickselect

You should be familiar with the algorithm for kth smallest. A quick version is as follows: given a sequence of n elements, we want to find the kth smallest one. At each step of the algorithm, we choose a pivot at random and partition the rest of the elements to the left and to the right of it, depending on whether they are smaller or larger. Depending on which of the generated sequences has to contain the element we are looking for, we recurse on one of them.

For good cost bounds, we are interested in making sure that the randomization introduces a constantly decreasing input size at each level of the recursion. What is the probability that at any given level, the sequence passed down to the next level is going to be less than $\frac{3}{4}$ of the original sequence?

Now that we have found the probability, what is the expected number of rounds we have to do to decrease a sequence to $< \frac{3}{4}$ of its original size?

4.2 Quicksort

Quisort is very similar to quickselect, except that we recurse on both of the generated subsequences. For the analysis of the cost bounds of quicksort, we are interested in the number of comparisons that happen between all the elements. What is the probability that two elements of the input sequence, element with rank i and element with rank j, are compared?