Recitation 3 – *More Recurrences and Scan*

Parallel and Sequential Data Structures and Algorithms, 15-210 (Spring 2014)

January 28th, 2014

1 Announcements

- Lab 2 – *SkylineLab* has been released and is due next Monday, February 3rd. This lab is conceptually much more difficult than the previous one, so start early!
- Questions from lecture or homework?

2 Recurrences

Let's first get some more practice with recurrences.

2.1 Example 1

$$f(n) = f(n/4) + \Theta(\lg^2 n)$$

2.1.1 Brick Method.

At level i:

Problem Size	$n/4^i$
Node Cost	$\leq k_1 \lg^2(n/4^i) + k_2$
Number of Nodes	1

++++++++++++++++++
+++++++++++++++++
++++++++++++++++
+++++++++++++++
++++++++++++++
...

This may look root-dominated: the level costs get smaller as we go down the tree. However, they don't get smaller by a constant factor, since the $1/4^i$ is inside a logarithm. Thus, we are in a balanced situation and $f(n)$ is $O(d \cdot \lg^2 n) = O(\lg^3 n)$.

2.1.2 Tree Method.

From the chart above, we get this not-so-friendly looking summation:

$$f(n) = k_1 \sum_{i=0}^{\lg n} (\lg^2(n/4^i)) + k_2 \lg n$$

$$f(n) = k_1 \sum_{i=0}^{\lg n} (\lg n - \lg 4^i)^2 + k_2 \lg n$$

Solving this summation exactly seems daunting, but we can make some headway using asymptotic approximations. [1] The second term, $k_2 \lg n$, is clearly lower-order and so we can drop it. The highest-order term of the summand is going to be $O(\lg^2 n)$, since $\lg 4^i$ is a constant with respect to n. Since we are summing $\lg n$ terms, each on the order of $\lg^2 n$, we can guess that the result will be $\Theta(\lg^3 n)$. This doesn't seem sufficiently formal, so a good thing to do is check ourselves using the substitution method, now that we have a guess.

2.1.3 Substitution Method.

We will show that the solution to the recurrence is $O(\lg^3 n)$. To do this, we wish to prove that there exist k_1, k_2 such that for all $n > 1$,

$$f(n) \le k_1 \lg^3 n + k_2$$

Base Case: From the definition of Θ, we know that there exists $c_1 > 0$ such that $f(1) \le c_1$. This proves the base case of our theorem as long as $c_1 \le k_2$. Let's remember that so we can choose an appropriate k_2.

Inductive Case: We know there exists $c_2 > 0$ such that

$$f(n) \le f(n/4) + c_2 \lg^2 n$$

Apply the induction hypothesis.

$$\begin{aligned} f(n) &\le k_1 \lg^3(n/4) + k_2 + c_2 \lg^2 n \\ &= k_1(\lg^3 n - 3\lg 4 \lg^2 n + 3\lg^2 4 \lg n - \lg^3 4) + k_2 + c_2 \lg^2 n \end{aligned}$$

We rearrange this suggestively.

$$f(n) \le k_1 \lg^3 n + k_2 + k_1(-3\lg 4 \lg^2 n + 3\lg^2 4 \lg n - \lg^3 4) + c_2 \lg^2 n$$

We are done as long as

$$k_1(-3\lg 4 \lg^2 n + 3\lg^2 4 \lg n - \lg^3 4) + c_2 \lg^2 n \le 0$$

[1] If this seems a lot like the brick method, it is. The brick method is essentially a useful shortcut to the tree method that allows us to gain intuitive understanding of the behavior of the recurrence without explicitly solving the summation.

We need to find k_1 that makes this true, so let's solve the above for k_1.

$$\begin{aligned} k_1 &\geq \frac{c_2 \lg^2 n}{3 \lg 4 \lg^2 n - 3 \lg^2 4 \lg n + \lg^3 4} \\ &= \frac{c_2}{3 \lg 4 - 3 \lg^2 4 \lg^{-1} n + \lg^3 4 \lg^{-2} n} \end{aligned}$$

We can satisfy these constraints by setting $k_1 = c_2$ and $k_2 = c_1$.

Thus, the recurrence is $O(\lg^3 n)$. Indeed, it is also $\Theta(\lg^3 n)$. We could show this by flipping around the theorem for the substitution method and proving that it is $\Omega(\lg^3 n)$.

2.2 Example 2

$$f(n) = 2f(\sqrt{n}) + \Theta(1)$$

This is somewhat harder than the recurrences we've solved before, but let's give it a try. Note that 1 won't work as a base case here (why?), so we assume $f(2) \in \Theta(1)$ as the base case.

2.2.1 Brick Method.

This time, we can just draw the tree since it's a familiar one:

$$\begin{array}{c} + \\ ++ \\ ++++ \\ ++++++++ \\ ++++++++++++++++ \\ \cdots \end{array}$$

The cost is dominated by the leaves, since the node cost is constant but the number of nodes grows exponentially. Thus, $f(n)$ is $O(\text{cost}_d) = O(2^d)$, where d is the number of levels in the tree.

But, how many levels are there? This is equivalent to asking how many times you can take the square root of a number before you get to 2.

$$n^{1/2^i} = 2$$

That's a messy exponent, so let's keep taking logs and hope it gets better.

$$1/2^i \lg n = \lg 2$$

$$i \lg 1/2 + \lg \lg n = \lg \lg 2$$

Noting that $\lg 1/2 = -1$ and $\lg \lg 2 = 0$, this gives

$$i = \lg \lg n$$

So, $d \approx \lg \lg n$, and $f(n) \in O(2^{\lg \lg n}) = O(\lg n)$.

2.2.2 Tree Method.

At level i:

Problem Size	$n^{1/2^i}$
Node Cost	$\leq k$
Number of Nodes	2^i

Given the chart and our calculation of d, we can fairly easily get the summation

$$f(n) = k \sum_{i=0}^{\lg\lg n} 2^i$$

$$f(n) = k(2^{\lg\lg n+1} - 1)$$

$$f(n) = k(2 \cdot 2^{\lg\lg n} - 1)$$

$$f(n) = k(2\lg n - 1) \in O(\lg n)$$

2.2.3 Substitution Method.

We want to show that there exist k_1, k_2 such that

$$f(n) \leq k_1 \lg n + k_2$$

Base Case: We know $f(2) \leq c_1$ for some c_1, so we need $c_1 \leq k_1 \lg 2 + k_2 = k_1 + k_2$.

Inductive Case: Apply the inductive hypothesis to our assumption.

$$f(n) \leq 2(k_1 \lg \sqrt{n} + k_2) + c_2$$

$$f(n) \leq 2(\frac{k_1}{2} \lg n + k_2) + c_2$$

$$f(n) \leq k_1 \lg n + 2k_2 + c_2$$

This works exactly if we set $k_2 = -c_2$. Does that work with our constraint from the base case?

$$c_1 \leq k_1 - c2$$

$$k_1 \geq c_1 + c_2$$

We can satisfy this constraint by setting $k_1 = c_1 + c_2$, so this completes the proof.

Again, we have shown only that the recurrence is $O(\lg n)$. We leave as an exercise the other direction required to show $\Theta(\lg n)$.

3 Scan

Yesterday, we covered the function `scan`. We'll recap the definition of `scan` briefly today, and show you how to solve interesting problems with it.

`scan` takes a function as one of its arguments. All of the text below makes the assumption that this function is *associative*. Recall the mathematical definition that a function f is said to be associative if and only if

$$\forall a \forall b \forall c. f(f(a,b),c) = f(a, f(b,c))$$

We also make the assumption that the initial value is a *left-identity* of the functional argument. Recall the mathematical definition that I is a left-identity of f if and only if

$$\forall a. f(I,a) = a$$

We don't need these assumptions in general, and we'll come back to a version of `scan` later that doesn't have them, but it's a cleaner way to start thinking about `scan` with these properties.

With the assumption that `f` is associative, `(scan f b)` is logically equivalent to `(iterh f b)` in the same way that `(reduce f b)` is logically equivalent to `(iter f b)`, but these functions differ in their span. Specifically, if `f` is a function that takes no more than a constant number of steps on all input, `(iterh f)` and `(iter f)` have both work and span in $O(n)$, whereas `reduce` and `scan` both have work in $O(n)$ and span in $O(\lg n)$.

It's worth noting that while `reduce` and `scan` are highly parallel, unlike `iter` and `iterh`, they pay the price by having slightly less general types.

3.1 Note on Terminology

If f is a function and I is a relevant identity for f, we'll often say "f-scan" to mean scan f I For example, a "+-scan" is scan (op +) 0

3.2 Recap

If $\texttt{s} = \langle 1, 6, 3, -2, 9, 0, -4 \rangle$, then

`(scan Int.min Int.maxInt s)` yields the following:

$(\langle \texttt{Int.maxInt}, 1, 1, 1, -2, -2, -2 \rangle, -4)$

Remember that in the result, location i stores the "sum" of the values at locations **before** i in the original sequence. There is a variant of `scan` called `scanI` which sums the values at locations before and including i.

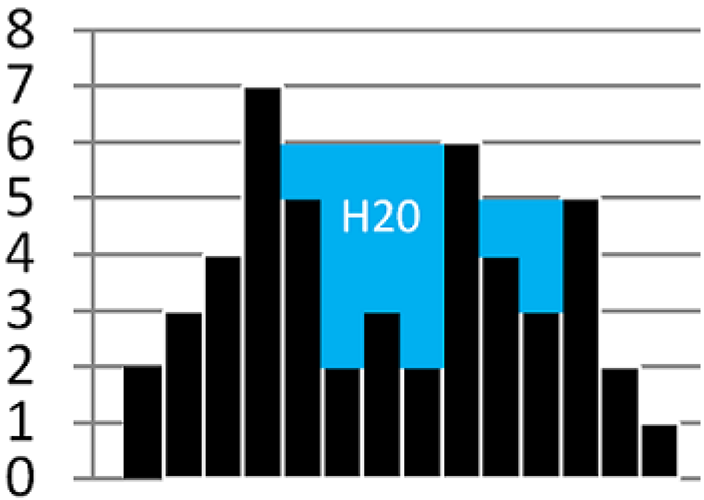
3.3 Example Uses of Scan

At first glance, `scan` seems to offer not much that isn't already available through `reduce`. With clever choices of associative functions, though, `scan` can be used to compute some surprising things efficiently in parallel.

3.3.1 Histogram

Consider the following problem:

Given a histogram, if we were to pour water over it, how much water (in terms of area) would it hold? For simplicity we will represent a histogram as a sequence of non-negative integers. For example the histogram shown below is represenented by the sequence $\mathtt{s} = \langle 2, 3, 4, 7, 5, 2, 3, 2, 6, 4, 3, 5, 2, 1 \rangle$, and holds 15 units of water.



Any ideas on how we might solve this problem?

The idea is to single out one bar b_i. If we know the maximum of the bar heights to the left of b_i ($\max_l$) and the maximum of the bar heights to the right of b_i ($\max_r$), given that $\max_l > \text{height}(b_i)$ and $\max_r > \text{height}(b_i)$ then the water b_i will hold above it is $\min(\max_l, \max_r) - \text{height}(b_i)$.

When confronted with a problem like this, a good technique is to divide up the problem into smaller subproblems, each of which can be easily solved with `scan`, `map` and/or `reduce`. Here's one way to divide it up, which directly follows the text in the previous paragraph.

1. For each bar b_i, calculate $\max_l$ and $\max_r$.
2. For each bar b_i, let $w_i = \min(\max_l, \max_r) - \text{height}(b_i)$ if $\max_l > \text{height}(b_i)$ and $\max_r > \text{height}(b_i)$, or $w_i = 0$ otherwise.
3. Sum all of the w_i.

Step 3 can be done with a `reduce`, and steps 1 and 2 can be done for each b_i in parallel, but separately calculating, for example, $\max_r$ for b_i and b_{i+1} will redo a lot of work, since these two bars share many of the same bars to their right. How can we complete steps 1 and 2 in parallel without duplicating work? Let's rearrange the above list slightly.

1. Calculate $\max_l$ for each b_i.
2. Calculate $\max_r$ for each b_i.
3. For each b_i, find $h_i = \min(\max_l, \max_r)$.
4. For each b_i, let $w_i = \max(h_i - b_i, 0)$.
5. Sum all of the w_i.

Note that we have split the previous step 2 into 2 steps, 3 and 4. This is starting to look more tractable. Let's take each step, assuming `hist` is a sequence of integers representing the histogram.

1. We've more or less already seen how to do this with `scan`. We just change the code above to use `Int.max` instead of `Int.min`:

   ```
   val (lHeights, _) = scan Int.max 0 hist
   ```

2. This is similar to step 1, except we want to take the max of all of the values to the right. We can still use `scan` for this, we just want to do a scan on the reversed list. Let's assume we have a function `rev` that reverses a sequence:

   ```
   val (rHeightsRev, _) = scan Int.max 0 (rev hist)
   ```

3. The phrase "for each" should imply that a `map` is in order. But we want to map over two sequences, `lHeights` and `rev rHeightsRev` (note that we need to reverse `rHeightsRev` again since it was generated by a scan over a reversed sequence.) For this, we can use `map2`:

   ```
   val heights = map2 Int.min lHeights (rev rHeightsRev)
   ```

4. We define a function `nonNegative` as follows:

   ```
   fun nonNegative (maxHeight, thisHeight) =
      Int.max (maxHeight - thisHeight, 0)
   ```

   Step 4 can then be accomplished by mapping this function over `heights` and the original histogram:

   ```
   map2 nonNegative heights hist
   ```

5. Finally, we do a reduce to add all of these heights:

   ```
   reduce op+ 0 (map2 nonNegative heights hist)
   ```

Defining rev and putting it all together gives us the complete SML code for the histogram filling problem:

```
fun rev s =
    let val n = length s
    in tabulate (fn i => nth s (n - i - 1)) n
    end

fun histogramFill (hist : int seq) =
    let
      val (lHeights, _) = scan Int.max 0 hist
      val (rHeightsRev, _) = scan Int.max 0 (rev hist)
      val heights = map2 Int.min lHeights (rev rHeightsRev)

      fun nonNegative (maxHeight, thisHeight) =
          Int.max (maxHeight - thisHeight, 0)
    in
      reduce op+ 0 (map2 nonNegative heights hist)
    end
```

3.3.2 Matching Parentheses

We can use scan to solve the parenthesis matching problem that we went over two weeks ago. The idea is that we first map each open parenthesis to 1 and each close parenthesis to -1. We then do a $+$-scan on this integer sequence. The elements in the sequence returned by scan exactly correspond how many unmatched parentheses there are in that prefix of the string. This is very much like the sequential algorithm we looked at in recitation, but scan lets us parallelize it! Recall that the parentheses are matched if and only if the counter never goes negative and is 0 at the end. We can check the first condition using a reduce over the sequence returned by scan, and the second by simply looking at the final value returned by scan.

For example:

$$\langle (,), (, (,),),) \rangle$$

becomes

$$\langle 1, -1, 1, 1, -1, -1, -1 \rangle$$

and then $+$-scan gives

$$(\langle 0, 1, 0, 1, 2, 1, 0 \rangle, -1)$$

and then fails, because the counter went negative at some point indicating an imbalance. The SML code for the parenthesis matching problem using scan is as follows:

```
fun match s =
    let
      fun paren2int OPAREN = 1
        | paren2int CPAREN = ~1
```

```
    val C = map paren2int s
    val (S,total) = scan (op+) 0 C
    val SOME(maxint) = Int.maxInt
  in
    (reduce Int.min maxint S) >= 0  andalso total = 0
  end
```