Recitation 5 — Graph Representations, StSeqs, Staging

Parallel and Sequential Data Structures and Algorithms, 15-210 (Fall 2012)

September 26, 2012

1 Announcements

- How did Assignment 3 go?
- Assignment 4 is out: You can do Task 2.1-2.3. Guy will cover shortest paths in class tomorrow.
- Questions about homework, lecture, life?

2 Friends of Friends

Suppose we have just founded a new social networking site and naturally we have represented the network of friends as a graph. We have are using the adjacency table representation of the graph. As discussed in class, an adjacency table is a table of sets where each element in the table maps a vertex to a set of its neighbors.

Suppose we want to find friends of our friends. These would be the neighbors of our neighbors. That is, find all vertices that have a distance of two from a source vertex v. We will need to introduce another function `tabulate` that can be found in the Table ADT. It allows us to perform a `map` over a set, but it creates a table, where the keys are from the set and values are the result of applying the map function. (Recall, there is no `map` for sets, because the results of the map may not result in unique values.) `Table.tabulate` has signature:

```
Table.tabulate : (Table.key -> 'a) -> Table.set -> Table.table
```

For example if we had a set $S = \{3,4,5\}$ and applied

```
val squares = Table.map (fn x => x*x) S
```

then `squares` $= \{3 \mapsto 9, 4 \mapsto 16, 5 \mapsto 25\}$.

To find friends of friends we might write:

```
1  type graph = Set.set Table.table
2  fun FoF (G : graph) v =
3  let
4    fun neighbors v = Table.find G v
5    val Ngh = neighbors v
6    val NghNgh = Table.tabulate neighbors Ngh
7  in
8    Table.reduce Table.Set.union {} NghNgh
9  end
```

In Line 5 we find the neighbors of v and in Line 6 we find the neighbors' neighbors, which now is a table mapping each neighbor to its neighbors. To flatten these sets of neighbors we do a `reduce` using set union for combining so that duplicates are removed. Note that the friends of our friends might not include our friends.

For example, let's say we start with the graph:

$$G = \{\text{“a”} \mapsto \{\text{“b”}, \text{“c”}\}, \text{“b”} \mapsto \{\text{“d”}\}, \text{“c”} \mapsto \{\text{“d”}, \text{“e”}\}, \text{“d”} \mapsto \{\text{“a”}\}, \text{“e”} \mapsto \{\}\}$$

Now we have:

```
 FoF G "a"
⇒  {"d","e"}
```

with intermediate results:

```
Ngh = {"b","c"}
NghNgh = {"b" ↦ {"d"}, "c"  ↦ {"d","e"}}
```

The `reduce` with `union` will combine the two sets for the given result.

3 Representing Graphs With Array Sequences

When graphs are implemented using tables and sets, the cost of finding a vertex in the table is $(\log n)$ work and span (recall that $n = |V|$). We can improve the asymptotic performance of certain algorithms if the search is constant work. An array sequence is an obvious data structure that has lookup with constant work.

To take advantage of the faster access of sequences we can use sequences to represent graphs. But the drawback is that this representation is less general, requiring the vertex names be restricted to integers in a fixed range. This restriction can become inconvenient in graphs that change dynamically but typically is fine for static graphs.

We refer to a graph $G = (V, E)$ where $V = \{0, 1, \ldots, n-1\}$ as an *integer labeled* (IL) graph. For such an IL graph, an α `vertexTable` can be represented as a sequence of length n with the values stored at the appropriate indices. In particular, the table

$$\{(0 \mapsto a_0), (1 \mapsto a_1), \cdots, (n-1 \mapsto a_{n-1})\}$$

is equivalent to the sequence

$$\langle a_0, a_1, \cdots, a_{n-1} \rangle ,$$

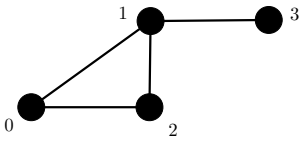
using standard reductions between sequences and sets. If we use an array representation of sequences, then this gives us constant work access to the values stored at vertices. We can also represent the set of neighbors of a vertex as an integer sequence containing the indices of those neighbors. Therefore, instead of using a

```
set table
```

to represent a graph we can use a

```
(int seq) seq.
```

For example, the following undirected graph:



would be represented as

$$G = \langle\langle 1,2\rangle, \langle 0,2,3\rangle, \langle 0,1\rangle, \langle 1\rangle\rangle.$$

The question is what are the costs of various operations on such a representation. Can you recall the costs We covered the costs for an adjacency table representation that we covered in class? What might they be using adjacency sequences?

	adj table		**adj seq**	
	work	*span*	*work*	*span*
`isEdge`$(G,(u,v))$	$O(\log n)$	$O(\log n)$	$O(d_G^+(u))$	$O(\log d_G^+(u))$
`map over all edges`	$O(m)$	$O(\log n)$	$O(m)$	$O(1)$
`map over neighbors of` v	$O(\log n + d_G^+(v))$	$O(\log n)$	$O(d_G^+(v))$	$O(1)$
$d_G^+(v)$	$O(\log n)$	$O(\log n)$	$O(1)$	$O(1)$

4 Reviewing `stseq`

Single-threaded sequences introduced in the previous lecture supported constant-work for each lookup and element update, as long as the operations always occurred on the most recent version of the sequence. Although the costs of these operations can be greater when applied to an earlier version of a sequence, the operations still work correctly. But, as single-threaded sequences use mutation, they appear functional only to a sequential observer and are unsafe for parallel observers. They include, however, a parallel update operation `inject`, so we can still get parallel performance.

	Work	Span
`fromSeq(S) :` α `seq` $\rightarrow$ α `stseq` Converts from a regular sequence to a stseq.	$O(\|S\|)$	$O(1)$
`toSeq(ST) :` α `stseq` $\rightarrow$ α `seq` Converts from a stseq to a regular sequence.	$O(\|S\|)$	$O(1)$
`nth ST` i `:` α `stseq` $\rightarrow$ `int` $\rightarrow$ α Returns the i^{th} element of ST. Same as for seq.	$O(1)$	$O(1)$
`update` (i,v) S `:` `(int` $\times$ α`)` $\rightarrow$ α `stseq` $\rightarrow$ α `stseq` Replaces the i^{th} element of S with v.	$O(1)$	$O(1)$
`inject` I S `:` `(int` $\times$ α`) seq` $\rightarrow$ α `stseq` $\rightarrow$ α `stseq` For each $(i,v) \in I$ replaces the i^{th} element of S with v.	$O(\|I\|)$	$O(1)$

4.1 Discussing complexity of `stseq` operations

```
1  let
2     a = ⟨2,5,1,7⟩
3  in
4        b = stseq.fromSeq a
5        c = update (0,3) b
6        d = update (1,2) c
7        d′ = update (3,12) c
8        e = nth c 5
9        f = nth d′ 5
10 end
11
```

Line 6: O(1)

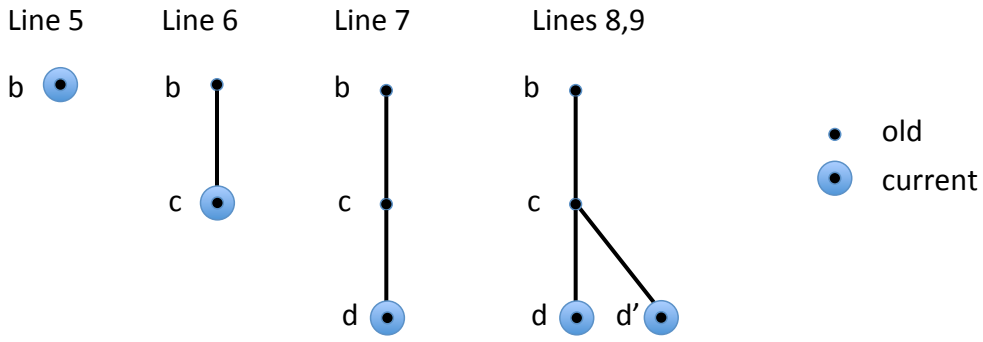
Line 7: O(possibly large number)

Line 8: O(large number)

Line 9: O(1)

Updating the current version takes constant work. However, updates can take a long time if the update is not made to the most recent version of the stseq. This is why the `update` from line 7 could also take a long time. Similarly `nth` can take a long time if not applied on the most recent version, as in line 8. When operating on the current version we can just look up the value in the current copy, which is up to date, as is the case in lines 6 and 9. When operating on an old version, however, we have to go back to the original copy and then check all the changes in the change log to see if any have modified the location we are asking about. This can be expensive. This is why updating and reading the current version is cheap ($O(1)$ work) while working with an old version is expensive.

The figure below illustrates which versions of the `stseq`s are current and old for each line of code above:



4.2 Marking my friends friend's with StSeqs

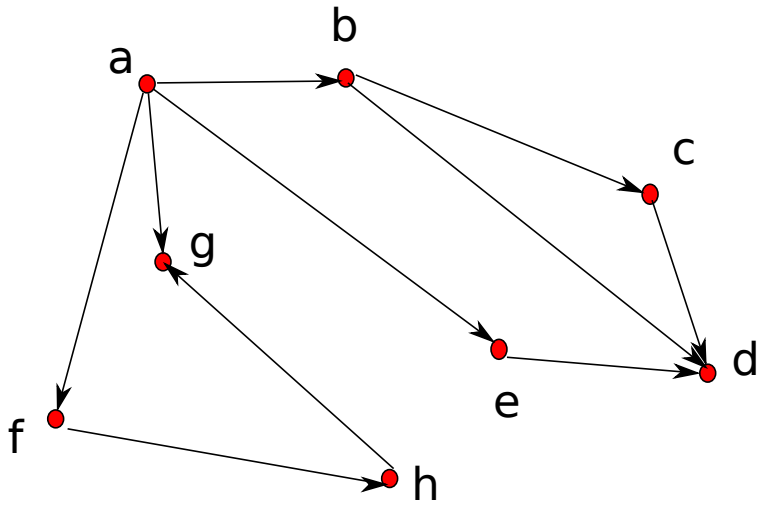
We started with an example of finding the neighbors of the neighbors of a vertex v in a graph represented as a `set table`. The expensive operation was taking the union of all the sets of neighbors. Instead, here we represent the graph using array sequences, and use a Boolean `stseq` to mark which vertices are the neighbors of neighbors.

```
1  type graph = (int Seq.seq) Seq.seq

2  fun FoF (flags : bool StSeq.stseq) (G : graph) (v : int) =
3  let
4    fun neighbors v = Seq.nth G v
5    val NghNgh = Seq.flatten(Seq.map neighbors (neighbors v))
6    val I = Seq.map (fn i ⇒ (i, true)) NghNgh
7  in
8    StSeq.inject I flags
9  end
```

In this code the `flags` argument is a Boolean `stseq` of length $|V|$. The sequence `NghNgh` contains all neighbors' neighbors of v, but can have repeats. For each, it writes `true` to that vertex' index in the `flags` sequence using `inject`. Now the expensive operation is flatten which uses $O(\sum_{u \in N^+(v)} d^+(u))$ work and has $O(\log N^+(v))$ span.

5 Explanation of Assignment 04



- Task 2.1: You have to make a graph based on a sequence of directed edges, using any representation you want.
- Task 2.3: For vertex a, `out_neighbors` should return $\langle b, e, f, g \rangle$
- Task 2.4: For vertex a, what should `make_asp` return? Which edges would never appear in any shortest path?

  Edges (c,d) and (h,g) would never appear in any shortest path.

  How many shortest paths are there from a to d? There are two shortest paths from a to d, and both should be represented in the resulting graph.
- Task 2.5: For vertex d, `report` should return $\langle \langle a, b, d \rangle, \langle a, e, d \rangle \rangle$

- Task 3.1: In light of what has been discussed so far, think of how you should represent the `thesaurus`.

6 Staging

Assignment 4 asks you to use staging. Let's now quickly work a simple staging example. Suppose you want to implement function that returns the nth largest value from an unsorted int sequence. Here is the type:

```
val nthLargest :  int Seq.seq -> int -> int
```

Non-stageable implementation:

```
fun nthLargest s n =  Seq.nth (Seq.sort Int.compare s)  n
```

To see why this isn't stageable, let's move the second argument to an inner function:

```
fun nthLargest s = fn n => Seq.nth (Seq.sort Int.compare s)  n
```

What happens if we apply this function to some sequence? e.g. `<4,5,3,7>`:

```
- val app = nthLargest <4,5,3,7>;
val app = fn n => Seq.nth (Seq.sort Int.compare <4,5,3,7>) n
: int -> int
```

Now every time we call `app` on some number, the `sort` function is invoked.

Instead, we can precompute this.

Q: How would we write a stageable version of `nthLargest`?

A:

```
fun nthLargestStaged  s =
let
    val presorted = Seq.sort String.compare s
in
    fn n => Seq.nth presorted n
end
```

Notice how the `sort` computation is no longer *guarded* by a function binding (`fn`). This means that when we apply it to a sequence e.g. `<4,5,3,7>`, we get

```
- val app = nthLargest <4,5,3,7>;
val app = fn n => Seq.nth <3,4,5,7> n : int -> int
```

Note that we can rewrite the last line of our staged function to

```
Seq.nth presorted
```

which is exactly equivalent, but the staging is clearer to see with the explicit binding.

Q: And how do we use the staged function?

A: We first call it without the n-argument:

```
val f = nthLargestStaged S;

val i = f 0;
val j = f 1;
val k = f 2;

etc...
```

Note: This can be called a high-order function, as it returns another function.

Q: Can you give examples of staged functions in our library?

A: map, scan, reduce, are *curried* functions. Staged functions are curried functions, but staging implies that the first stage performs some serious computation. In our library, there are not really staged functions, because it does not include complex algorithms.