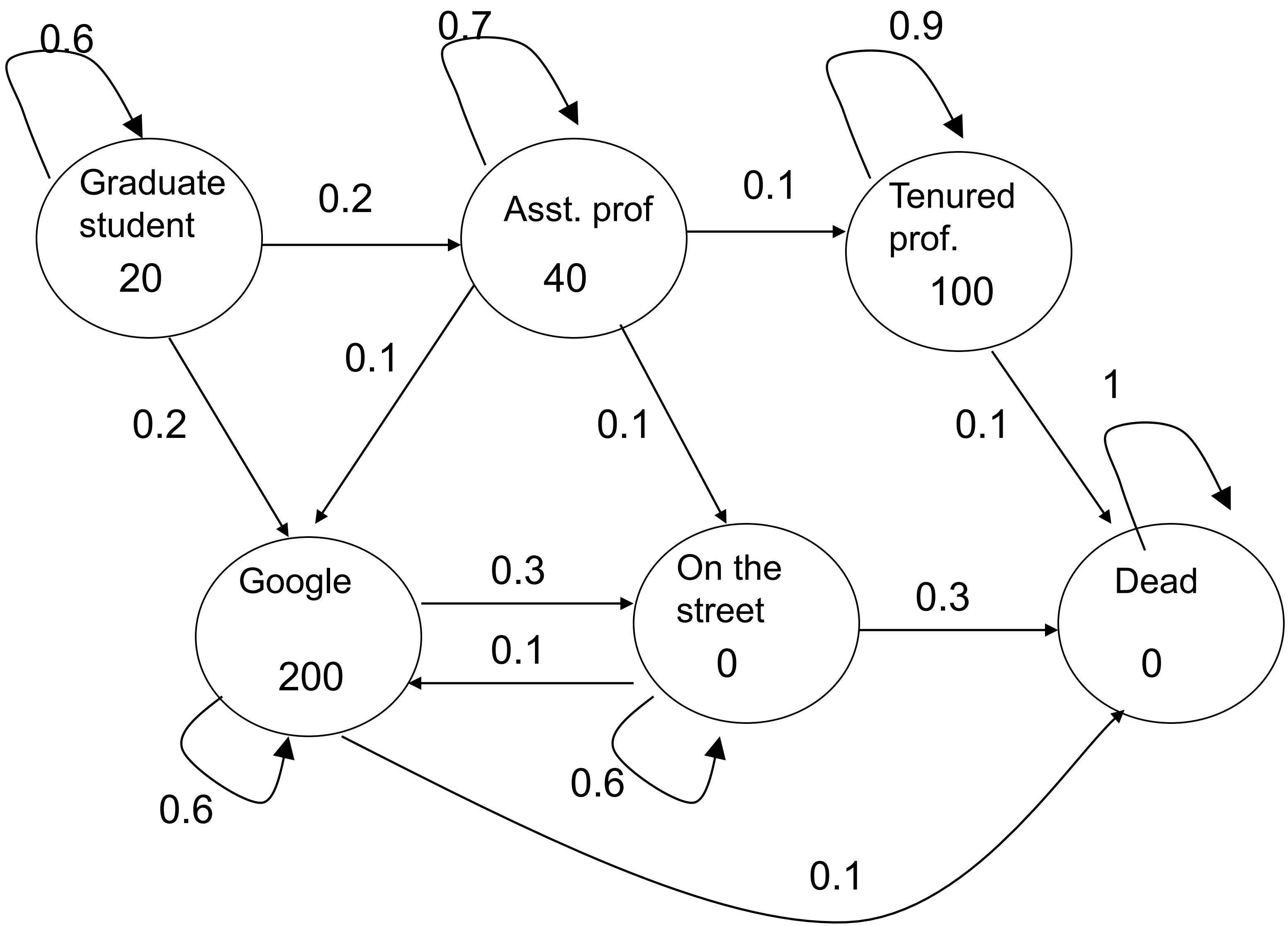
10-601
Machine Learning

Markov decision processes (MDPs)

What's missing in HMMs

- HMMs cannot model important aspects of agent interactions:
  - No model for rewards
  - No model for actions which can affect these rewards
- These are actually issues that are faced by many applications:
  - Agents negotiating deals on the web
  - A robot which interacts with its environment

Example: No actions



Formal definition of MDPs

- A set of states $\{s_1 \dots s_n\}$
- A set of rewards $\{r_1 \dots r_n\}$
- A set of actions $\{a_1 .. a_m\}$
- Transition probability

One reward for each state

Number of actions could be larger than number of states

$$P_{i,j}^{k} = P(q_{t+1} = s_j \mid q_t = i \;\&\; h_t = a_k)$$

Questions

- What is my expected pay if I am in state *i*
- What is my expected pay if I am in state *i* and perform action *a*?

Solving MDPs

- No actions: Value iteration
- With actions: Value iteration, Policy iteration

Value computation

- An obvious question for such models is what is combined expected value for each state
- What can we expect to earn over our life time if we become Asst. prof.?
- What if we go to industry?

Before we answer this question, we need to define a model for future rewards:

• The value of a current award is higher than the value of future awards

- Inflation, confidence

- Example: Lottery

Discounted rewards

- The discounted rewards model is specified using a parameter γ
- Total rewards = current reward +

γ (reward at time t+1) +

γ^2 (reward at time t+2) +

…

γ^k (reward at time t+k) + …

infinite sum

Discounted rewards

- The discounted rewards model is specified using a parameter γ
- Total rewards = current reward +

γ (reward at time t+1) +

γ^2 (reward at time t+2) +

Converges if $0<\gamma<1$

k) + …

infinite sum

Determining the total rewards in a state

- Define $J^*(s_i)$ = expected discounted sum of rewards when starting at state s_i
- How do we compute $J^*(s_i)$?

Factors expected pay for all possible transitions for step *i*

$$J^*(s_i) = r_i + \gamma X$$
$$= r_i + \gamma(p_{i1}J^*(s_1) + p_{i2}J^*(s_2) + \cdots p_{in}J^*(s_n))$$

How can we solve this?

Computing j*(s_i)

$$J^*(s_1) = r_1 + \gamma(p_{11}J^*(s_1) + p_{12}J^*(s_2) + \cdots p_{1n}J^*(s_n))$$

$$J^*(s_2) = r_2 + \gamma(p_{21}J^*(s_1) + p_{22}J^*(s_2) + \cdots p_{2n}J^*(s_n))$$

$$J^*(s_n) = r_n + \gamma(p_{n1}J^*(s_1) + p_{n2}J^*(s_2) + \cdots p_{nn}J^*(s_n))$$

- We have n equations with n unknowns
- Can be solved in closed form

Iterative approaches

- Solving in closed form is possible, but may be time consuming.
- It also doesn't generalize to non-linear models
- Alternatively, this problem can be solved in an iterative manner
- Lets define $J^t(s_i)$ as the expected discounted rewards after t steps
- How can we compute $J^t(s_i)$?

$$J^1(S_i) = r_i$$

$$J^2(S_i) = r_i + \gamma\left(\sum_k p_{i,k} J^1(s_k)\right)$$

$$J^{t+1}(S_i) = r_i + \gamma\left(\sum_k p_{i,k} J^t(s_k)\right)$$

Iterative approaches

We know how to solve this!

Let's fill the dynamic programming table

- Lets define $J^t(s_i)$ as the expected discounted awards after t steps

But wait …

This is a never ending task!

$$J^2(S_i) = r_i + \gamma\left(\sum_k p_{i,k} J^1(s_k)\right)$$

$$J^{t+1}(S_i) = r_i + \gamma\left(\sum_k p_{i,k} J^t(s_k)\right)$$

When do we stop?

$$J^1(S_i) = r_i$$

$$J^2(S_i) = r_i + \gamma\left(\sum_k p_{i,k} J^1(s_k)\right)$$

$$J^{t+1}(S_i) = r_i + \gamma\left(\sum_k p_{i,k} J^t(s_k)\right)$$

Remember, we have a converging function

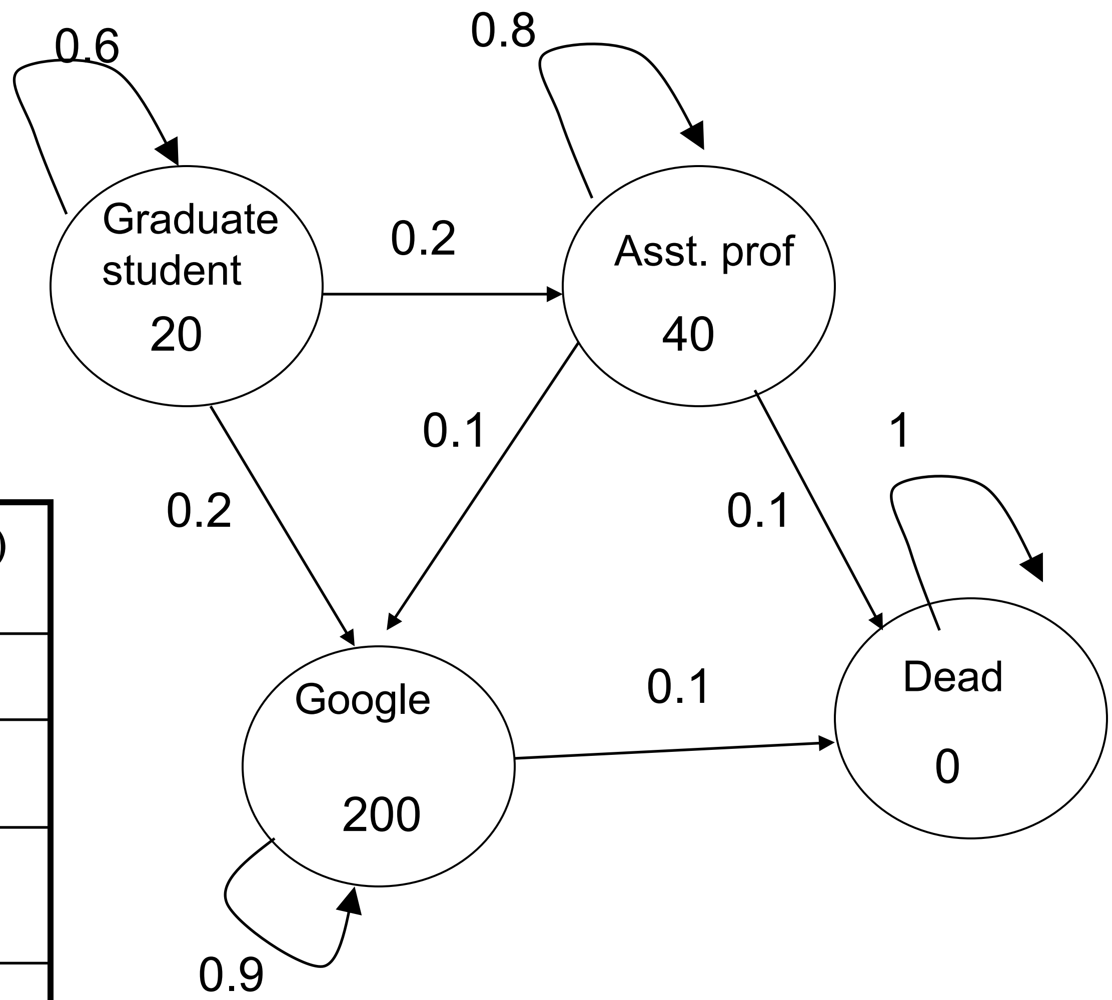
We can stop when $|J^{t-1}(s_i) - J^t(s_i)|_\infty < \varepsilon$

Infinity norm selects maximal element

Example for γ=0.9

$J^2(Gr)=20+0.9*(0.6*20+0.2*40+0.2*200)$

t	$J^t(Gr)$	$J^t(P)$	$J^t(Goo)$	$J^t(D)$
1	20	40	200	0
2	74	87	362	0
3	141	135	493	0
4	209	182	600	0



Solving MDPs

- No actions: Value iteration √

- With actions: Value iteration, Policy iteration

Adding actions

A Markov Decision Process:

- A set of states $\{s_1 \ldots s_n\}$
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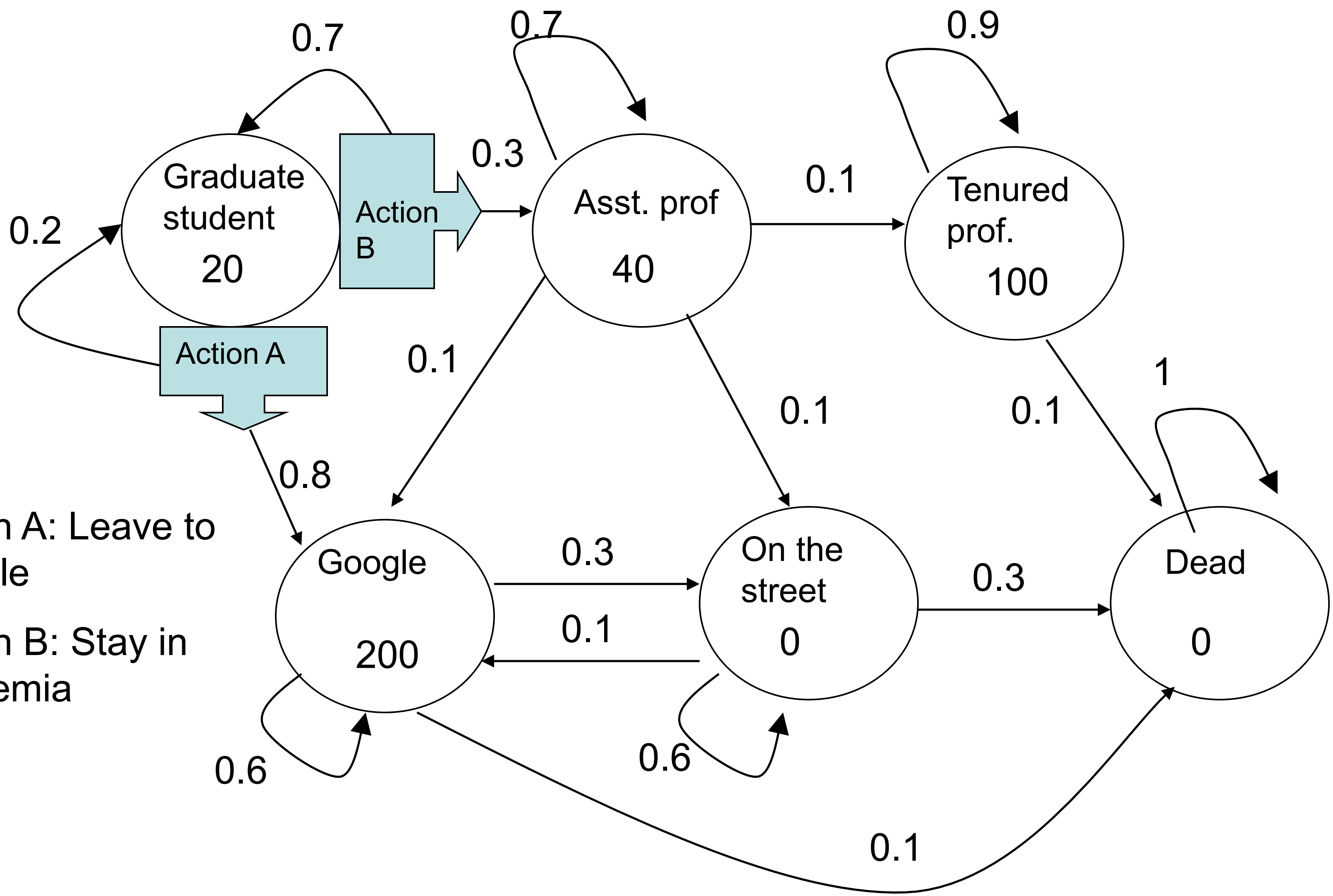
- Transition probability

$$P_{i,j}^{k} = P(q_{t+1} = s_j \mid q_t = i \,\&\, h_t = a_k)$$

Example: Actions



Action A: Leave to Google

Action B: Stay in academia

Questions for MDPs

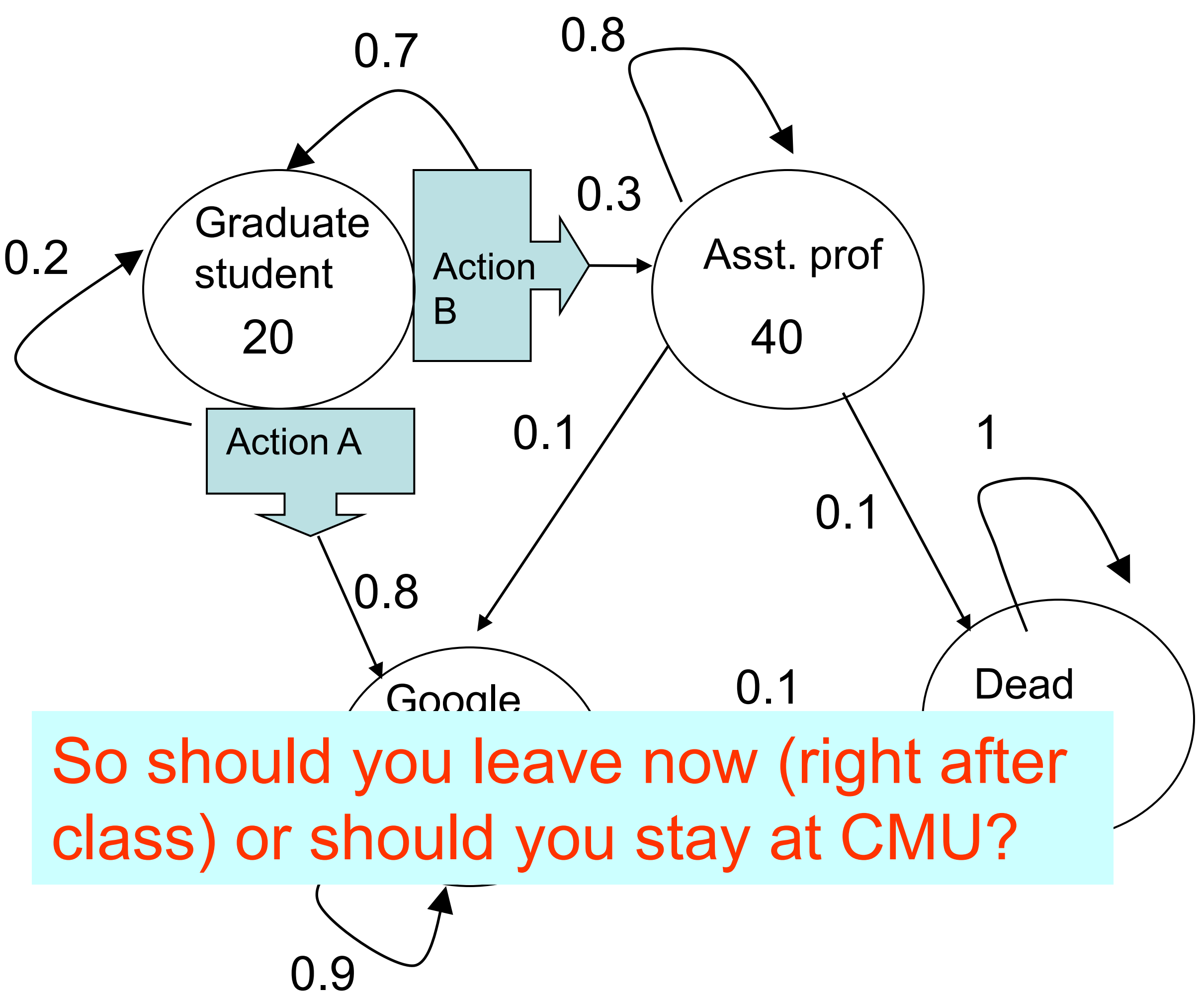
- Now we have actions
- The question changes to the following:

Given our current state and the possible actions, what is the best action for us in terms of long term payment?

Example: Actions

Action A: Leave to Google

Action B: Stay in academia



So should you leave now (right after class) or should you stay at CMU?

Policy

- A policy maps states to actions
- An optimal policy leads to the highest expected returns
- Note that this does not depend on the start state

Gr	B
Go	A
Asst. Pr.	A
Ten. Pr.	B

Solving MDPs with actions

- It could be shown that for every MDP there exists an optimal policy (we won’t discuss the proof).
- Such policy guarantees that there is no other action that is expected to yield a higher payoff

Computing the optimal policy: 1. Modified value iteration

- We can compute it by modifying the value iteration method we discussed.
- Define p^k_{ij} as the probability of transitioning from state i to state j when using action k
- Then we compute:

Use probabilities associated with action k

$$J^{t+1}(S_i) = \max_k r_i + \gamma\left(\sum_j p^k_{i,j} J^t(s_j)\right)$$

Also known as Bellman's equation

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Run until convergence

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- Then we compute:

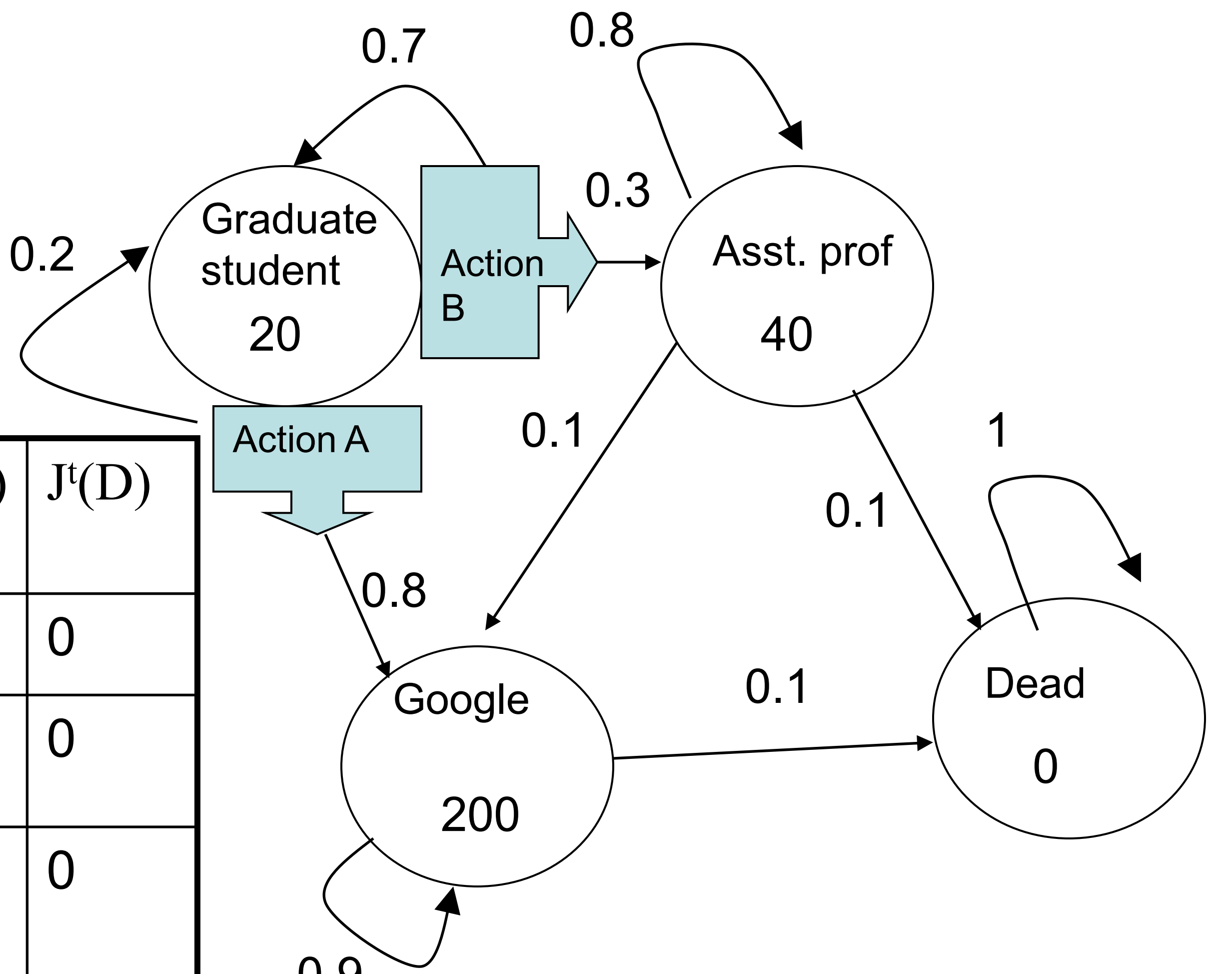
$$J^{t+1}(S_i) = \max_k r_i + \gamma\left(\sum_j p^k_{i,j} J^t(s_j)\right)$$

- When the algorithm converges, we have computed the best outcome for each state
- We associate states with the actions that maximize their return

Value iteration for γ=0.9

J2(Gr) = 20+ 0.9*Max {
0.2*20+0.8*200,
0.7*20+0.3*40}

t	$J^t(Gr)$	$J^t(P)$	$J^t(Goo)$	$J^t(D)$
1	20	40	200	0
2	168(A) 51(B)	87	362	0
3	311(A) 120(B)	135	493	0
4	431(A) 189(B)	182	600	0



Computing the optimal policy: 2. Policy iteration

- We can also compute optimal policies by revising an existing policy.
- We initially select a policy at random (mapping from states to actions).
- We re-compute the expected long term reward at each state using the selected policy
- We select a new policy using the expected rewards and iterate until converges

Policy iteration: algorithm

- Let $\pi_t(s_i)$ be the selected policy at time t

1. Randomly chose π_0 ; set t = 0
2. For each state s_i compute $J^*(s_i)$, the long term expected reward using policy π_t .
3. Set $\pi_t(s_i)$ = $\max_k r_i + \gamma \left(\sum_j p_{i,j}^k J^*(s_j) \right)$
4. Convergence? Yes: output policy. No: t = t + 1, go to 2.

Policy iteration: algorithm

- Let $\pi_t(s_i)$ be the selected policy at time t

1. Randomly chose π_0 ; set t = 0
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Can be computed using $J^*(s_i)$ for all states

Once the policy is fixed we are back to rewards only models, so this can be computed using value iteration

Value iteration vs. policy iteration

- Depending on the model and the information at hand:

  - If you have a good guess regarding the optimal policy then policy iteration would converge much faster

  - Similarly, if there are many possible actions, policy iteration might be faster

  - Otherwise value iteration is a safer way

Demo

- http://www.cs.cmu.edu/~awm/rlsim/

What you should know

- Models that include rewards and actions
- Value iteration for solving MDPs
- Policy iteration

Partially Observed Markov Decision Processes (POMDPs)

- Same model as MDP except we do not observe the states we are in.
- Thus, we have a distribution over states
- There is an initial distribution for states (initial belief)
- Once we reach a new state and receive a reward we can re-compute a new belief regrading the possible set of states

Example

- If we see 1, we can be in any of several locations.
- However, based on past and future observations we can increase a decrease our belief at a given state

1	1	1
3	1	2
1	2	1

POMDPs can be solved by extending the MDP methods to solve for a belief state vector rather than for the original single state MDP