Problem 1
Determine asymptotically tight bounds for the following recurrences.

(a) \( T(n) = T(n/6) + T(n/3) + T(n/2) + n \)

We use the recursion-tree method:

The summation shows that \( n \cdot \log_6 n < T(n) < n \cdot \log_2 n \), which implies that \( T(n) = \Theta(n \cdot \lg n) \).

(b) \( T(n) = T(n - 1) + n \)

\[
T(n) = T(n - 1) + n \\
= T(n - 2) + (n - 1) + n \\
= T(n - 3) + (n - 2) + (n - 1) + n \\
\vdots \\
= T(1) + 2 + 3 + \ldots + (n - 1) + n \\
= T(1) + \frac{(n - 1) \cdot (n + 2)}{2} \\
= \Theta(n^2)
\]

(c) \( T(n) = T(n - 1) + 1/2^n \)

\[
T(n) = T(n - 1) + 1/2^n \\
= T(n - 2) + 1/2^{n-1} + 1/2^n \\
= T(n - 3) + 1/2^{n-2} + 1/2^{n-1} + 1/2^n \\
\vdots \\
= T(1) + 1/2^2 + 1/2^3 + \ldots + 1/2^{n-1} + 1/2^n \\
= T(1) + 1/2 - 1/2^n = \Theta(1)
\]
(d) $T(n) = T(\sqrt{n}) + 1$

For convenience, we assume that $n = 2^{2^k}$, for some natural value $k$.

\[
T(n) = T(\sqrt{2^{2^k}}) + 1 = T(2^{2^k-1}) + 1
\]
\[
= T(\sqrt{2^{2^k-1}}) + 1 + 1 = T(2^{2^k-2}) + 2
\]
\[
= T(\sqrt{2^{2^k-2}}) + 1 + 2 = T(2^{2^k-3}) + 3
\]
\[
\ldots
\]
\[
= T(2^{2^k-k}) + k
\]
\[
= T(2) + k
\]
\[
= \Theta(k)
\]

Note that $k = \lg \lg n$, which implies that $T(n) = \Theta(\lg \lg n)$.

(e) $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$

We assume for convenience that $n = 2^k$ and $T(4) = 4$, and use induction to prove the following equality:

\[T(2^k) = 2^k \cdot k.\]

This equality holds for $k = 1$:

\[T(2^1) = T(4) = 4 = 2^1 \cdot 1,
\]

and the induction step is as follows:

\[
T(2^{k+1}) = \sqrt{2^{2^{k+1}}} \cdot T(\sqrt{2^{2^{k+1}}}) + 2^{2^{k+1}}
\]
\[
= 2^k \cdot T(2^k) + 2^{2^{k+1}}
\]
\[
= 2^k \cdot (2^k \cdot k) + 2^{2^{k+1}}
\]
\[
= (2^k)^2 \cdot k + 2^{2^{k+1}}
\]
\[
= 2^{2k+1} \cdot k + 2^{2^{k+1}}
\]
\[
= 2^{2^{k+1}} \cdot (k + 1)
\]

Note that $k = \lg \lg n$, which implies that $T(n) = \Theta(n \cdot \lg \lg n)$. 

2