Bayes optimal classifier
Naïve Bayes

Machine Learning – 10701/15781
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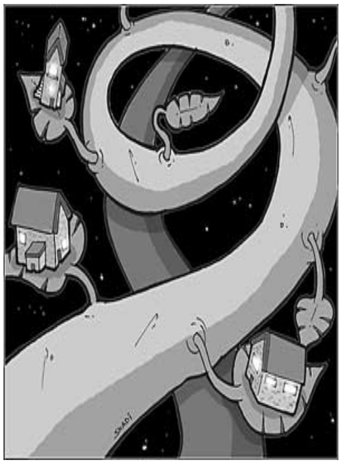
Machine learning for apartment hunting



- Now you've moved to Pittsburgh!!

  And you want to find the **most overall satisfying** apartment for you to **move in:**

  square-ft., # of bedroom, distance to campus, rent, …

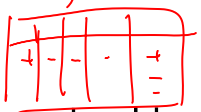
Living area (ft^2)	# bedroom	Rent ($)	Yes/No
230	1	600	yes
506	2	1000	yes
433	2	1100	no
109	1	500	no
…			
150	1	500	?
270	1.5	1200	?

Classification

- **Learn**: h:**X** ↦ Y
  - ☐ **X** – features
  - ☐ Y – target classes

(Rent, Ames, dist ...)

1/0



- Suppose you know P(Y|**X**) exactly, how should you classify?
  - ☐ Bayes classifier:

$\arg\max_y p(Y|X)$

$X_1 \, X_2 \cdots X_n \mid Y=1$, $Y=0$ $P(X|Y)$

$h(y|x) = P(X|y)P(Y)$.

if $P(x|y=1)P(Y=1) \geq P(X|y=0)P(y=0)$. then $Y=1$.

- **Why?**

Optimal classification

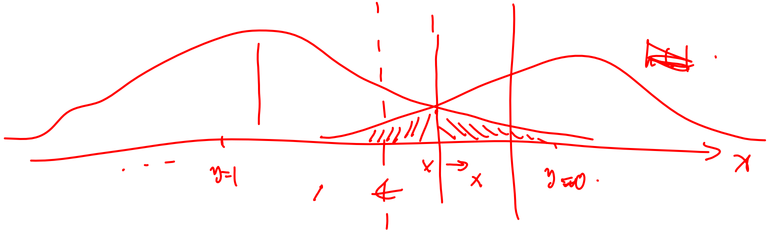
- **Theorem:** Bayes classifier h_{Bayes} is optimal!

$p(x|y=1)p(y=1) = p(x|y=0)p(y=0)$.

$p(x|y)p(y)$.

  - ☐ That is $error_{true}(h_{Bayes})) \leq error_{true}(h), \ \forall h(\mathbf{x})$

- **Proof**: $p(error) = \int_x p(error|x)p(x)dx$



Bayes Rule

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \sum_y P(X|Y)P(Y)$$

Which is shorthand for:

$$(\forall i,j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i) P(Y = y_i)}{P(X = x_j)}$$

How hard is it to learn the optimal classifier?

- Data =

Sky	Temp	Humid	Wind	Water	Forecst	EnjoySpt
Sunny	Warm	Normal	Strong	Warm	Same	Yes
Sunny	Warm	High	Strong	Warm	Same	Yes
Rainy	Cold	High	Strong	Warm	Change	No
Sunny	Warm	High	Strong	Cool	Change	Yes

- How do we represent these? How many parameters?
  - ☐ Prior, P(Y): $= \prod_k \theta_k^{\delta(y=k)} = \begin{cases} \theta_k & \text{if } y = k \end{cases}$
    - Suppose Y is composed of *k* classes

for K outcome & θ need K−1 numbers

  - ☐ Likelihood, P(**X**|Y):
    - Suppose **X** is composed of *n* binary features

y.	f_1	f_2		- - - - -		f_n	
	+	+	+	+	+	+	π_1.
1	+	-	+	+	+	+	
							0

- Complex model → High variance with limited data!!!

Conditional Independence

- X is **conditionally independent** of Y given Z, if the probability distribution governing X is independent of the value of Y, given the value of Z

$$(\forall i, j, k) P(\underline{X = i}|\underline{Y = j, Z = k}) = P(X = i|\underline{Z = k})$$

- e.g., $\underline{P(Thunder|Rain, Lightning) = P(Thunder|Lightning)}$

- Equivalent to:

$$\underline{P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)}$$

What if features are independent?

- Predict 10701Grade
- From two **conditionally Independent** features
  - ☐ HomeworkGrade X_1
  - ☐ ClassAttendance X_2

$$P(y)\,P(X|y)$$

$$\downarrow$$

$$\cancel{P(X|y)} \quad P(X_1\ X_2|y) \leftarrow$$

$$= P(X_1|y)\,P(X_2|y)$$

The Naïve Bayes assumption

- Naïve Bayes assumption:
  - ☐ Features are independent given class:

$$\underline{P(X_1, X_2|Y)} = P(X_1|X_2, Y)P(X_2|Y)$$
$$= P(X_1|Y)P(X_2|Y)$$

  - ☐ More generally:

$$\boxed{P(X_1 ... X_{\overset{6}{n}}|Y) = \prod_i P(X_i|Y)}$$

2^7
↓
12

- How many parameters now?
  - Suppose **X** is composed of *n* binary features

The Naïve Bayes Classifier

- Given:
  - ☐ Prior P(Y)
  - ☐ *n* conditionally independent features **X** given the class Y
  - ☐ For each X_i, we have likelihood $P(X_i|Y)$
- Decision rule:

$$\underline{y^* = h_{NB}(\mathbf{x})} = \arg\max_y P(y)P(x_1, \ldots, x_n \mid y)$$
$$= \arg\max_y P(y) \prod_i P(x_i|y)$$

- If assumption holds, NB is optimal classifier!

MLE for the parameters of NB

- Given dataset
  - ☐ Count(A=a,B=b) ← number of examples where A=a and B=b

#

- MLE for NB, simply:
  - ☐ Prior: P(Y=y) = $\frac{\#(y=1)}{\sum_k \#(y=k)}$
  - ☐ Likelihood: P(X$_i$=x$_i$|Y$_i$=y$_i$) = $\frac{\#(X=x_i,\ y_i=y_i)}{\#(y_i=y_i)}$

$P(X..y))$

$\times\ P(y)\ )$

$\sum_k \#(X_i=k\ y_i=y_i)$

Subtleties of NB classifier 1 – Violating the NB assumption

- Usually, features are not conditionally independent:

$$\boxed{P(X_1 \ldots X_n|Y) \neq \prod_i P(X_i|Y)}$$

- Actual probabilities P(Y|**X**) often biased towards 0 or 1
- Nonetheless, NB is the single most used classifier out there
  - ☐ NB often performs well, even when assumption is violated
  - ☐ [Domingos & Pazzani '96] discuss some conditions for good performance

Subtleties of NB classifier 2 – Insufficient training data

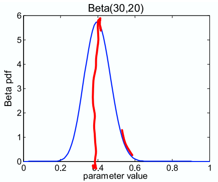
- What if you never see a training instance where X_1=a when Y=b?
    - ☐ e.g., Y={SpamEmail}, X_1={'Enlargement'}
    - ☐ P(X_1=a | Y=b) = 0 ←
- Thus, no matter what the values $X_2,\ldots,X_n$ take:
    - ☐ P(Y=b | X_1=a,$X_2,\ldots,X_n$) = 0

$\propto P(Y=b)\, P(x \cdots x_n \mid Y=b)$

$= P(Y=b) \prod_i P(X_i \mid Y=b) = 0$

- What now???

MAP for Beta distribution



$$P(\theta \mid \mathcal{D}) = \frac{\theta^{\beta_H+\alpha_{\overset{1}{H}}-1}(1-\theta)^{\beta_T+\alpha_{\overset{2}{T}}-1}}{B(\beta_H+\alpha_H,\beta_T+\alpha_T)} \sim Beta(\beta_H+\alpha_H, \beta_T+\alpha_T)$$

$\alpha_1 = \#(y=1)$ $\alpha_2 = \#(y=0)$

$P(y) = \begin{cases}\theta \\ 1-\theta\end{cases} = \theta^{\delta(y=1)}(1-\theta)^{1-\delta}$

- MAP: use most likely parameter:

$$\widehat{\theta} = \arg\max_{\theta} P(\theta \mid \mathcal{D}) = \frac{\alpha_1+\beta_1-1}{\alpha_1+\beta_1-1+\alpha_0+\beta_0-1}$$

$P(\theta) \propto \theta^{\beta_1-1}(1-\theta)^{\beta_0-1}$

$\frac{\beta_0}{\beta_1+\beta_0}$

$N \to \infty$

$\theta_{ML} = \frac{\alpha_1}{\alpha_1+\alpha_0}$

$m = \beta_1+\beta_2$

- Beta prior equivalent to extra thumbtack flips
- As $N \to \infty$, prior is "forgotten"
- **But, for small sample size, prior is important!**

Bayesian learning for NB parameters – a.k.a. smoothing

- Dataset of *N* examples
- Prior
  - ☐ “distribution” Q(X$_i$,Y), Q(Y)
  - ☐ *m* “virtual” examples

$m_{i/y}$

$$p(\pi_i^{k/Y}) = \mathrm{Beta}(\alpha_i^{k/Y}, \alpha_i^{k/Y})$$

$$P(\theta) = \mathrm{Beta}(\beta_1, \beta_2)$$

$$\beta_1 + \beta_2 = m$$

$$\alpha_i^{1/y} + \alpha_i^{0/y} = m_{i/y}$$

- MAP estimate
  - ☐ $P(X_i|Y_{=1}) = \dfrac{(\#X_i = x_i)\cancel{f}(y_i = 1) + \alpha_i^{1/Y=1}}{\cancel{D}\,\#(y=1) + m_{i/y=1}}$

$$P(y) = \frac{\#(y=1) + \beta_1}{N + m}$$

$p(X_i|y)\,P(y))$

- Now, even if you never observe a feature/class, posterior probability never zero

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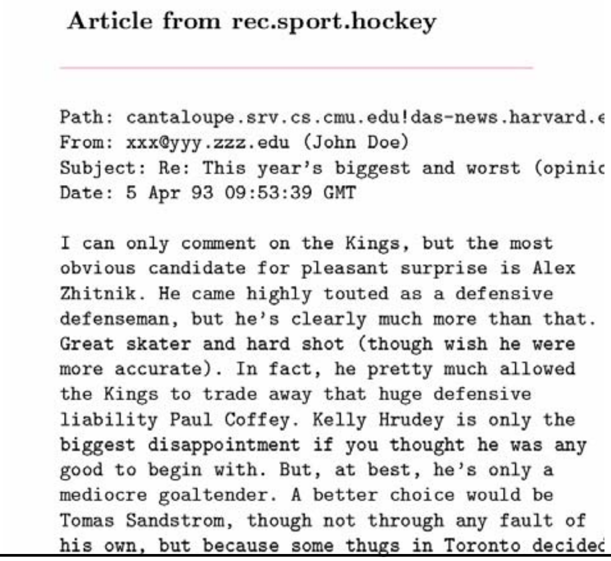
Text classification

- Classify e-mails
  - ☐ Y = {Spam,NotSpam}
- Classify news articles
  - ☐ Y = {what is the topic of the article?}
- Classify webpages
  - ☐ Y = {Student, professor, project, …}
- What about the features **X**?
  - ☐ The text!

Features **X** are entire document – X_i for i^{th} word in article



Article from rec.sport.hockey

```
Path: cantaloupe.srv.cs.cmu.edu!das-news.harvard.e
From: xxx@yyy.zzz.edu (John Doe)
Subject: Re: This year's biggest and worst (opinic
Date: 5 Apr 93 09:53:39 GMT

I can only comment on the Kings, but the most
obvious candidate for pleasant surprise is Alex
Zhitnik. He came highly touted as a defensive
defenseman, but he's clearly much more than that.
Great skater and hard shot (though wish he were
more accurate). In fact, he pretty much allowed
the Kings to trade away that huge defensive
liability Paul Coffey. Kelly Hrudey is only the
biggest disappointment if you thought he was any
good to begin with. But, at best, he's only a
mediocre goaltender. A better choice would be
Tomas Sandstrom, though not through any fault of
his own, but because some thugs in Toronto decidec
```

NB for Text classification

- P(**X**|Y) is huge!!!
  - ☐ Article at least 1000 words, $\mathbf{X}=\{X_1,\ldots,X_{1000}\}$ $\in 10000^{1000}$
  - ☐ X_i represents i^{th} word in document, i.e., the domain of X_i is entire vocabulary, e.g., Webster Dictionary (or more), 10,000 words, etc.

$P(X_1|y) =$

- NB assumption helps a lot!!!
  - ☐ $P(X_i=x_i|Y=y)$ is just the probability of observing word x_i in a document on topic y

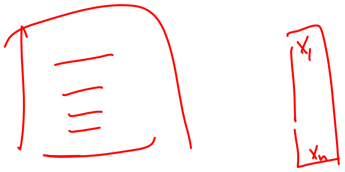
$$h_{NB}(\mathbf{x}) = \arg\max_y P(y) \underline{\prod_{i=1}^{LengthDoc} P(x_i|y)}$$

Bag of words model

- Typical additional assumption – **Position in document doesn't matter**: $P(X_i=x_i|Y=y) = P(X_k=x_i|Y=y)$
  - ☐ "Bag of words" model – order of words on the page ignored
  - ☐ Sounds really silly, but often works very well!

$$P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

When the lecture is over, remember to wake up the person sitting next to you in the lecture room.



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$$P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

in is lecture lecture next over person remember room sitting the the the to to up wake when you

Bag of Words Approach

the world of

TOTAL

▶ All About The Company
Global Activities
Corporate Structure
TOTAL's Story
Upstream Strategy
Downstream Strategy
Chemicals Strategy
TOTAL Foundation
Homepage

all about the **company**

Our energy exploration, production, and distribution operations span the globe, with activities in more than 100 countries.

At TOTAL, we draw our greatest strength from our fast-growing oil and gas reserves. Our strategic emphasis on natural gas provides a strong position in a rapidly expanding market.

Our expanding refining and marketing operations in Asia and the Mediterranean Rim complement already solid positions in Europe, Africa, and the U.S.

Our growing specialty chemicals sector adds balance and profit to the core energy business.

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

NB with Bag of Words for text classification

- Learning phase:
  - ☐ Prior P(Y)
    - Count how many documents you have from each topic (+ prior)
  - ☐ $P(X_i|Y)$
    - For each topic, count how many times you saw word in documents of this topic (+ prior)
- Test phase:
  - ☐ For each document
    - Use naïve Bayes decision rule

+ $\frac{1}{10{,}000}$

1

Smoothing

$$h_{NB}(\mathbf{x}) = \arg\max_y P(y) \prod_{i=1}^{\overset{K}{\cancel{LengthDoc}}} P(x_i|y)$$

Twenty News Groups results

Given 1000 training documents from each group
Learn to classify new documents according to
which newsgroup it came from

comp.graphics
comp.os.ms-windows.misc
comp.sys.ibm.pc.hardware
comp.sys.mac.hardware
comp.windows.x

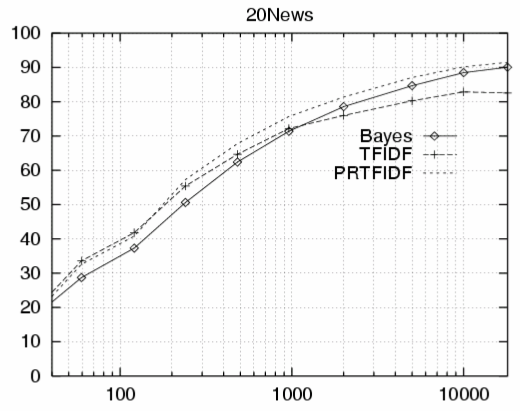
misc.forsale
rec.autos
rec.motorcycles
rec.sport.baseball
rec.sport.hockey

alt.atheism
soc.religion.christian
talk.religion.misc
talk.politics.mideast
talk.politics.misc
talk.politics.guns

sci.space
sci.crypt
sci.electronics
sci.med

Naive Bayes: 89% classification accuracy

Learning curve for Twenty News Groups



Accuracy vs. Training set size (1/3 withheld for test)

What if we have continuous X_i ?

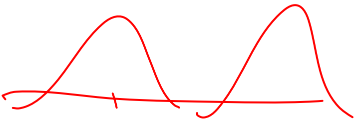
Eg., character recognition: X_i is i^{th} pixel

Gaussian Naïve Bayes (GNB):

$$\underline{P(X_i = x \mid Y = y_k)} = \underline{\frac{1}{\sigma_{ik}\sqrt{2\pi}} \, e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}}$$

Sometimes assume variance

- is independent of Y (i.e., σ_i),
- or independent of X_i (i.e., σ_k)
- or both (i.e., σ)



Estimating Parameters: Y discrete, X_i continuous

Maximum likelihood estimates:

jth training example

$$\hat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j \boxed{X_i^j} \underline{\delta(Y^j = y_k)}$$

δ(x)=1 if x true, else 0

$$\hat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k) - 1} \sum_j (X_i^j - \hat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

Example: GNB for classifying mental states

[Mitchell et al.]

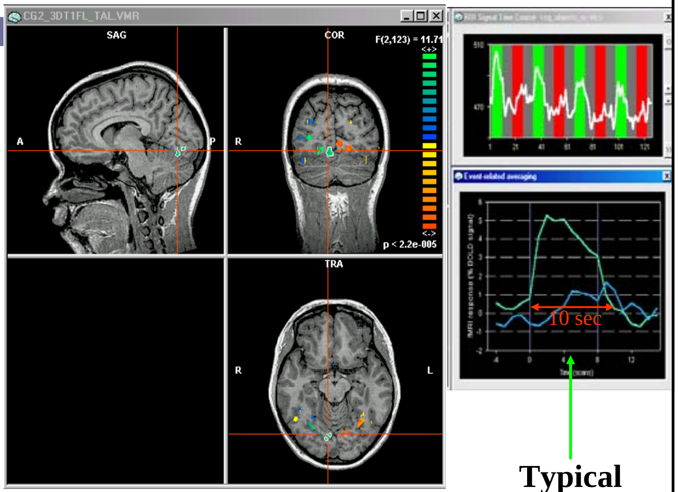
~1 mm resolution

~2 images per sec.

15,000 voxels/image

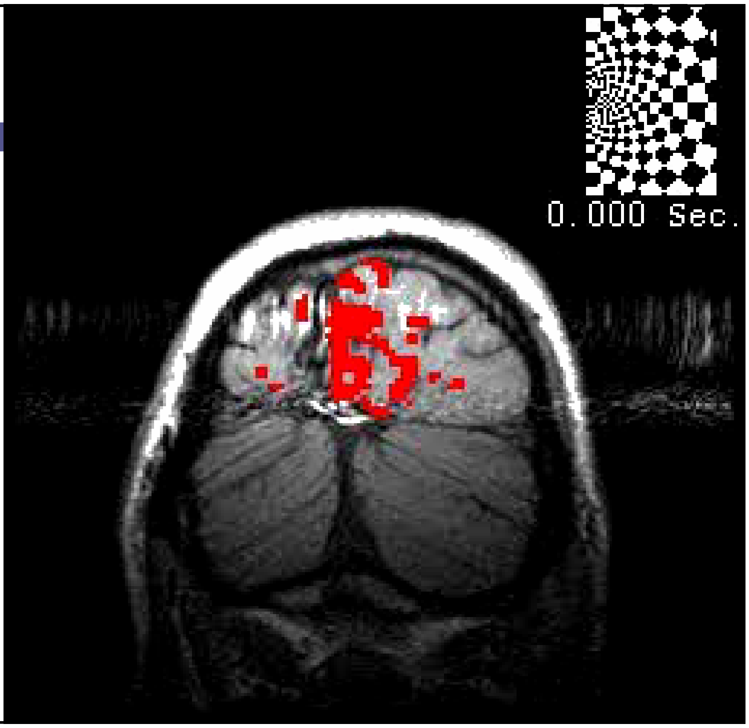
non-invasive, safe

measures Blood Oxygen Level Dependent (BOLD) response



Typical impulse response

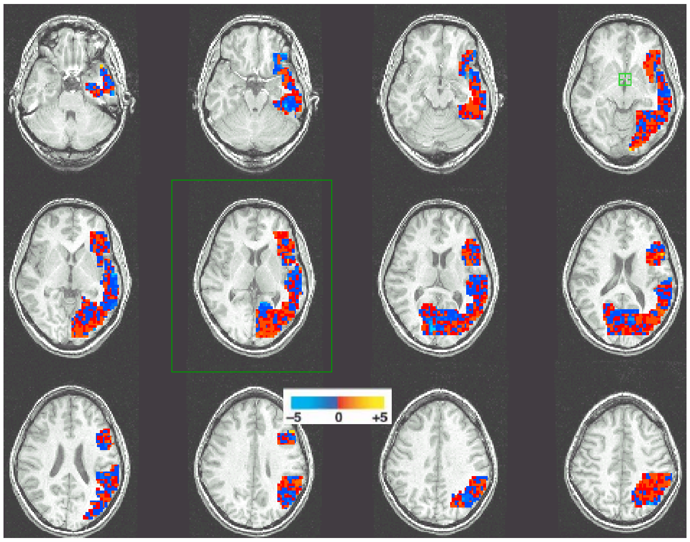
Brain scans can track activation with precision and sensitivity



[Mitchell et al.]

Gaussian Naïve Bayes: Learned $\mu_{voxel,word}$ P(BrainActivity | WordCategory = {People,Animal})

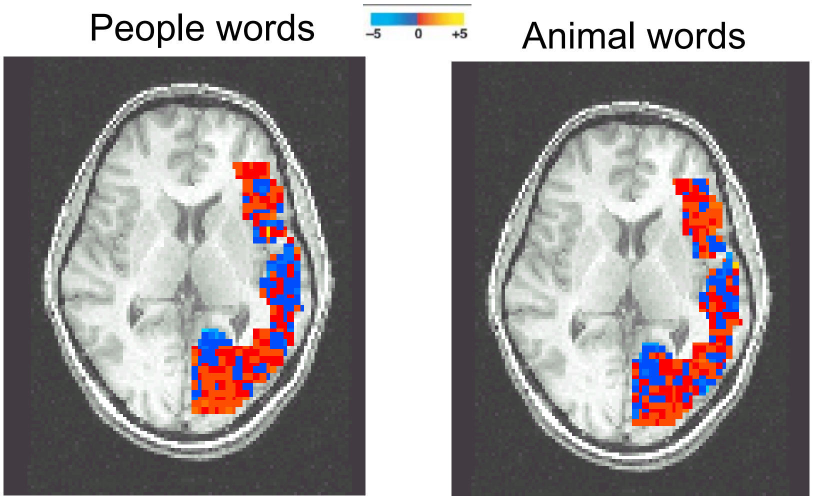
[Mitchell et al.]



Learned Bayes Models – Means for P(BrainActivity | WordCategory)

[Mitchell et al.]

Pairwise classification accuracy: 85%



People words

Animal words

What you need to know about Naïve Bayes

- Optimal decision using Bayes Classifier
- Naïve Bayes classifier
  - What's the assumption
  - Why we use it
  - How do we learn it
  - Why is Bayesian estimation important
- Text classification
  - Bag of words model
- Gaussian NB
  - Features are still conditionally independent
  - Each feature has a Gaussian distribution given class