15-462: Computer Graphics

Math for Computer Graphics

Topics for Today

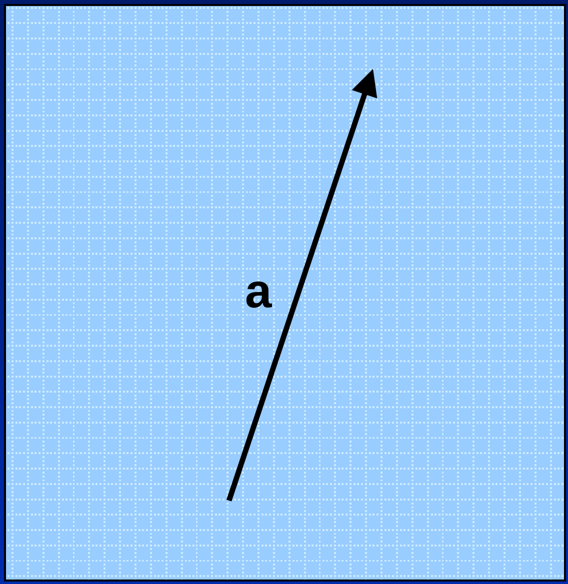
- Vectors
- Equations for curves and surfaces
- Barycentric Coordinates

Topics for Today

- Vectors
  - What is a vector?
  - Coordinate systems
  - Vector arithmetic
  - Dot product
  - Cross product
  - Normal vectors
- Equations for curves and surfaces
- Barycentric Coordinates

What is a vector?

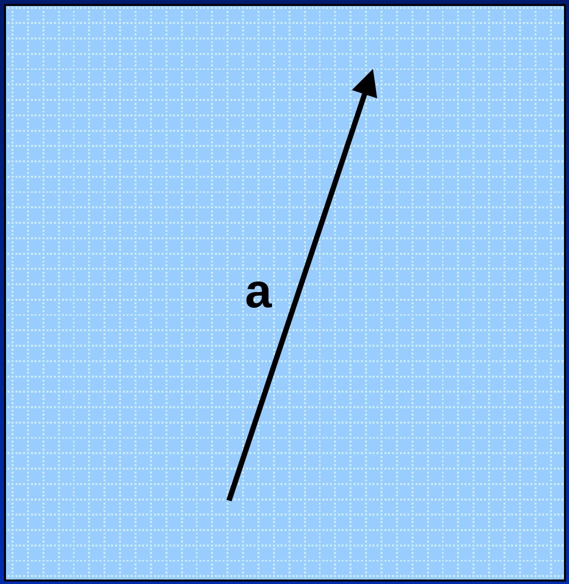
- A *vector* is a value that describes both a magnitude and a direction. We draw vectors as arrows, and name them with bold letters, e.g. **a**.



What is a vector?

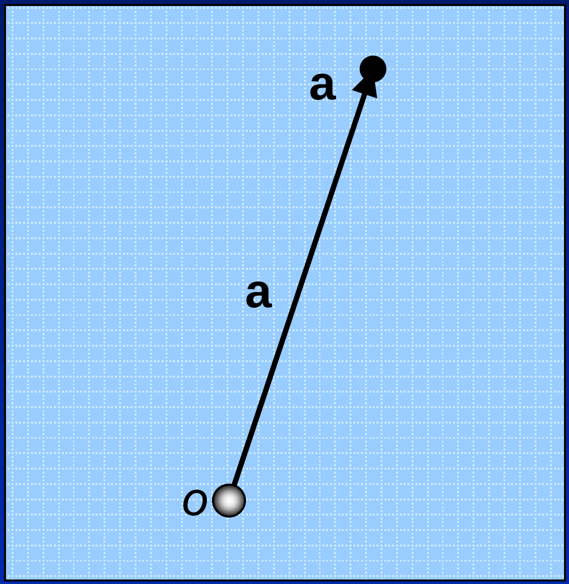
Some Definitions

- The *magnitude* of vector **a** is the scalar given by ||**a**||.
- A *unit vector* is any vector whose magnitude is one.
- The *zero vector*, **0**, has a magnitude of zero, and its direction is undefined.
- Two vectors are equal if and only if they have equal magnitudes and point in the same direction.



What is a vector?

- Vectors themselves contain no information about a starting point.
- We can interpret vectors as *displacements*, instructions to get from one point in space to another.
- We can also interpret vectors as *points*, but in order to do so, we must assume a particular *origin* as the starting point.

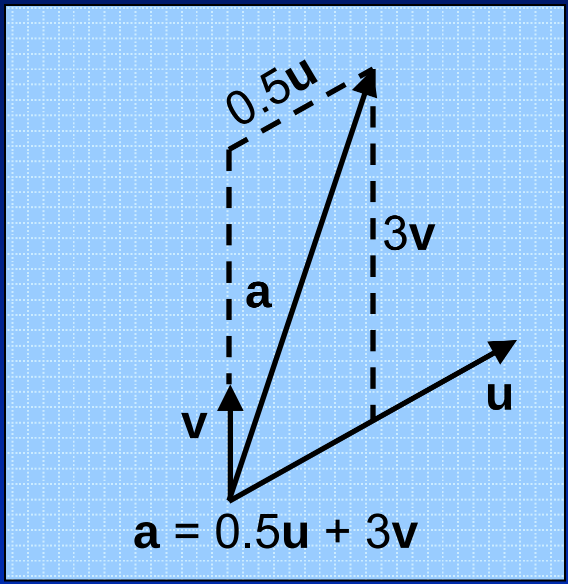


Coordinate systems

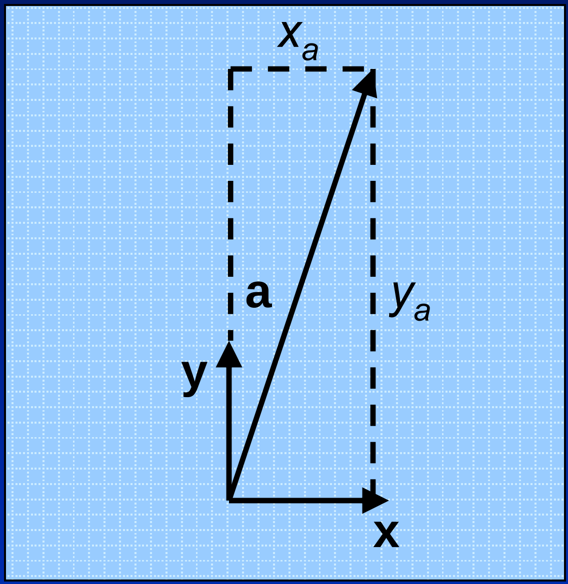
- A vector can be multiplied by a scalar to scale the vector's magnitude without changing its direction:

$$||k\mathbf{a}|| = k||\mathbf{a}||$$

- In 2D, we can represent any vector as a unique *linear combination*, or weighted sum, of any two non-parallel *basis vectors*.
- 3D requires three non-parallel, non-coplanar basis vectors.



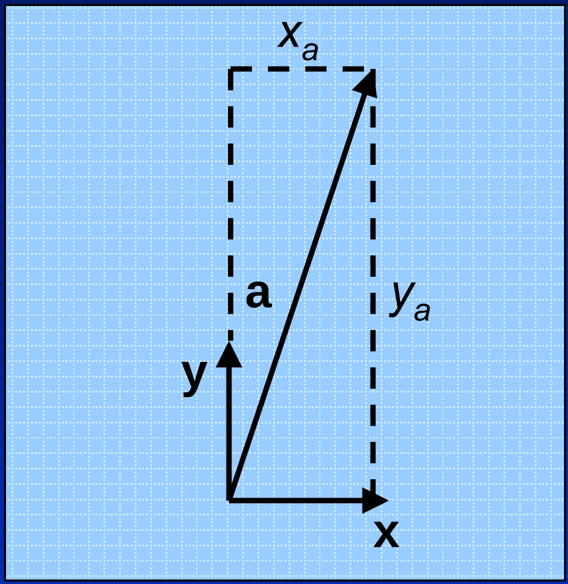
Coordinate systems



- Basis vectors that are unit vectors at right angles to each other are called *orthonormal*.
- The **x-y** *Cartesian* coordinate system is a special orthonormal system.
- Vectors are commonly represented in terms of their Cartesian coordinates:

$$\mathbf{a} = \left(x_a, y_a\right) \qquad \mathbf{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix} \qquad \mathbf{a}^T = \begin{bmatrix} x_a & y_a \end{bmatrix}$$

Coordinate systems



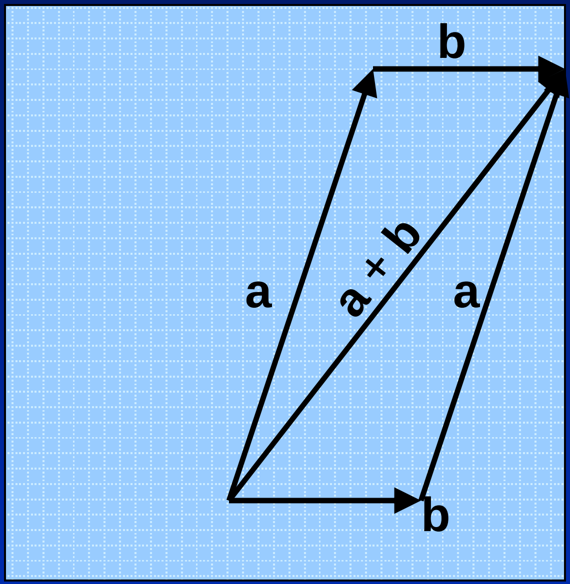
- Vectors expressed by orthonormal coordinates

$$\mathbf{a} = (x_a, y_a)$$

have the very useful property that their magnitudes can by calculated according to the Pythagorean Theorem:

$$||\mathbf{a}|| = \sqrt{x_a^2 + y_a^2}$$

Vector arithmetic



- To find the *sum* of two vectors, we place the tail of one to the head of the other. The sum is the vector that completes the triangle.
- Vector addition is commutative:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

Vector arithmetic

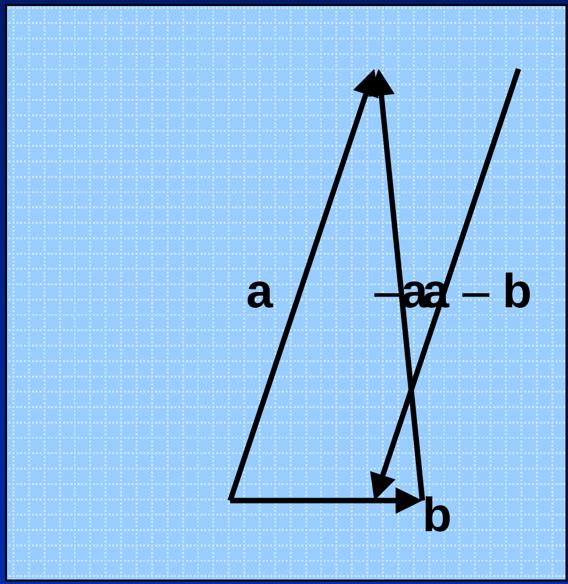
- We define the *unary minus* (negative) such that

$$-\mathbf{a} + \mathbf{a} = \mathbf{0}$$

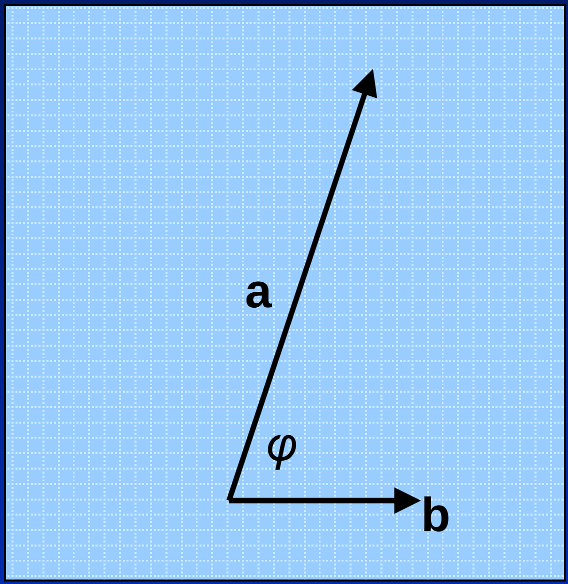
- We can then define *subtraction* as

$$\mathbf{a} - \mathbf{b} \equiv -\mathbf{b} + \mathbf{a}$$

- This gives the vector from the end of **b** to the end of **a** if both have the same origin.



Dot product



- We can multiply two vectors by taking the *dot product*.
- The dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos \varphi$$

  where φ is the angle between the two vectors.
- Note that the dot product takes two vectors as arguments, but it is often called the *scalar product* because its result is a scalar.

Dot product

Some cool properties:

- It's often useful in graphics to know the cosine of the angle between two vectors, and we can find it with the dot product:

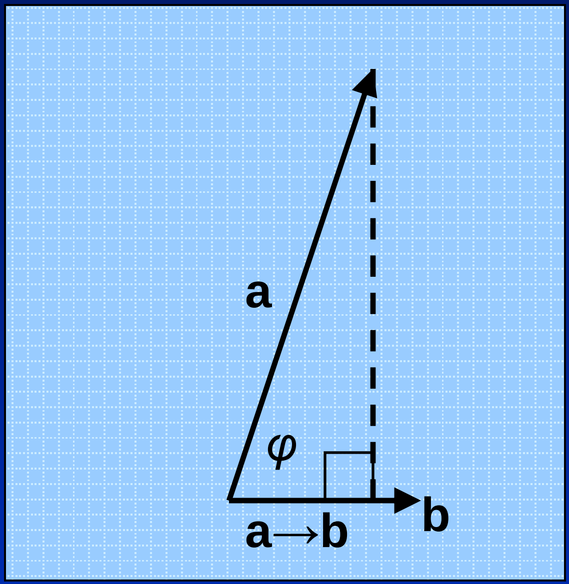
$$\cos \varphi = \mathbf{a} \cdot \mathbf{b} \,/\, (||\mathbf{a}||\ ||\mathbf{b}||)$$

- We can use the dot product to find the *projection* of one vector onto another. The scalar **a→b** is the magnitude of the vector **a** projected at a right angle onto vector **b**, and

$$\mathbf{a}\rightarrow\mathbf{b} = ||\mathbf{a}|| \cos \varphi = \mathbf{a} \cdot \mathbf{b} \,/\, ||\mathbf{b}||$$

- Dot products are commutative and distributive:

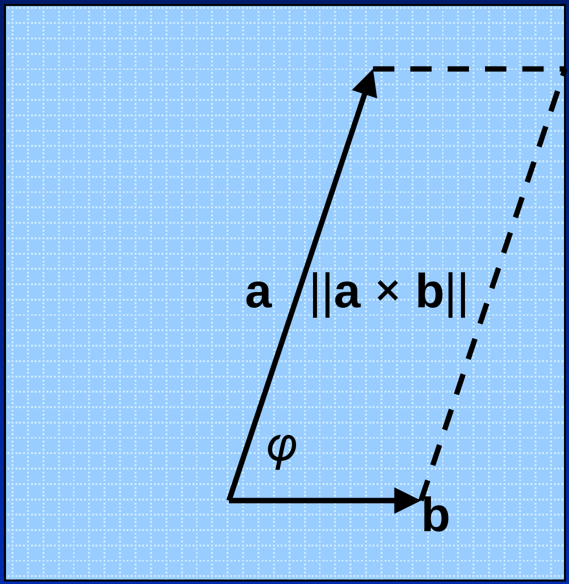
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$
$$(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$$



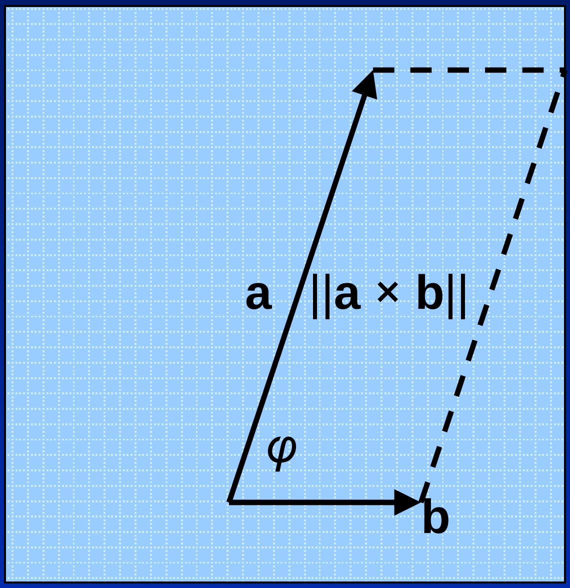
Cross product

- The *cross product* is another vector multiplication operation, usually used only for 3D vectors.
- The direction of **a** × **b** is orthogonal to both **a** and **b**.
- The magnitude is equal to the area of the parallelogram formed by the two vectors. It is given by

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\ \|\mathbf{b}\| \sin \varphi$$



Cross product



Some cool properties:

- Cross products are distributive:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$$

- Cross products are intransitive; in fact,

$$\mathbf{a} \times \mathbf{b} = -\, \mathbf{b} \times \mathbf{a}$$

- Because of the sine in the magnitude calculation, for all **a**,

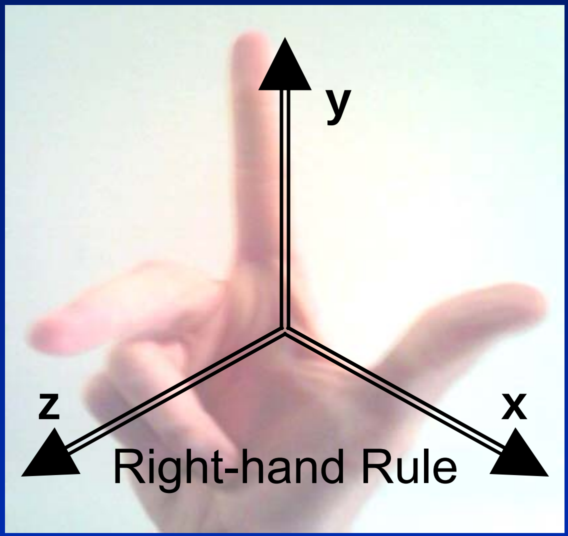
$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

- In **x-y-z** Cartesian space,

$$\mathbf{x} \times \mathbf{y} = \mathbf{z} \qquad \mathbf{y} \times \mathbf{z} = \mathbf{x} \qquad \mathbf{z} \times \mathbf{x} = \mathbf{y}$$

Cross product

- As defined on previous slides, the direction of the cross product is ambiguous.
- The *left-hand rule* and the *right-hand rule* distinguish the two choices.
- If **a** points in the direction of your thumb and **b** points in the direction of your index finger, **a × b** points in the direction of your middle finger.
- Of the two, the right-hand rule is the predominant convention.



Normal vectors

- A *normal vector* is a vector perpendicular to a surface. A *unit normal* is a normal vector of magnitude one.
- Normal vectors are important to many graphics calculations.
- If the surface is a polygon containing the points **a**, **b**, and **c**, one normal vector

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

- This vector points *into* the polygon if **a**, **b**, and **c** are arranged clockwise; it points outward if they are arranged counterclockwise.

Vectors

Chalkboard examples:

- Cartesian vector addition
- Cartesian dot product
- Cartesian cross product

Topics for Today

- Vectors
- Equations for curves and surfaces
  - Implicit equations
  - Parametric equations
- Barycentric Coordinates

Implicit equations

- *Implicit equations* are a way to define curves and surfaces.
- In 2D, a curve can be defined by

$$f(x,y) = 0$$

  for some scalar function f of x and y.
- In 3D, a surface can be defined by

$$f(x,y,z) = 0$$

  for some scalar function f of x, y, and z.

Implicit equations

- The function *f* evaluates to 0 at every point on the curve or surface, and it evaluates to a non-zero real number at all other points.
- Multiplying *f* by a non-zero coefficient preserves this property, so we can rewrite

$$f(x,y) = 0$$

$$\text{as } kf(x,y) = 0$$

for any non-zero *k*.

- The implied curve is unaffected.

Implicit equations

Chalkboard examples:

- Implicit 2D circle
- Implicit 2D line
- Implicit 3D plane

Implicit equations

- We call these equations “implicit” because although they imply a curve or surface, they cannot explicitly generate the points that comprise it.
- In order to generate points, we need another form…

Parametric equations

- *Parametric equations* offer the capability to generate continuous curves and surfaces.
- For curves, parametric equations take the form

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

- For 3D surfaces, we have

$$x = f(s,t) \quad y = g(s,t) \quad z = h(s,t)$$

Parametric equations

- The *parameters* for these equations are scalars that range over a continuous (possibly infinite) interval.
- Varying the parameters over their entire intervals smoothly generates every point on the curve or surface.

Implicit equations

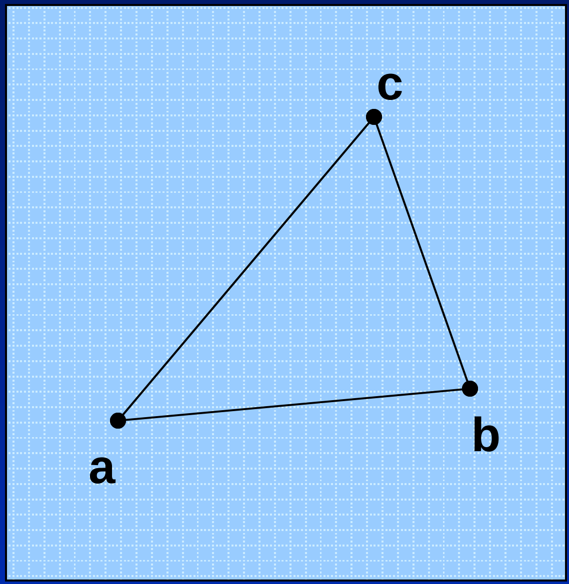
Chalkboard examples:

- Parametric 3D line
- Parametric sphere

Topics for Today

- Vectors
- Equations for curves and surfaces
- Barycentric Coordinates
  - Why barycentric coordinates?
  - What are barycentric coordinates?

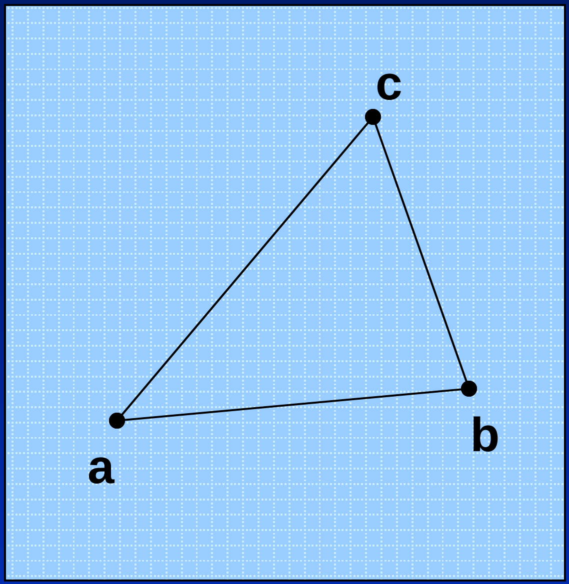
Why barycentric coordinates?



- Triangles are the fundamental primitive used in 3D modeling programs.
- Triangles are stored as a sequence of three vectors, each defining a vertex.
- Often, we know information about the vertices, such as color, that we'd like to interpolate over the whole triangle.

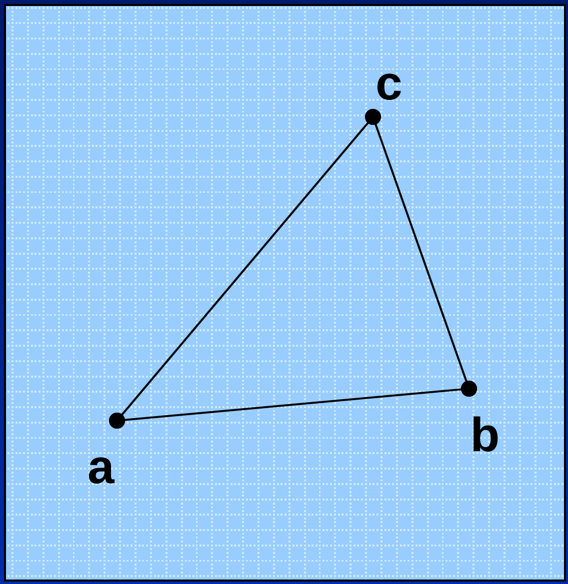
What are barycentric coordinates?

- The simplest way to do this interpolation is *barycentric coordinates*.
- The name comes from the Greek word *barus* (heavy) because the coordinates are weights assigned to the vertices.
- Point **a** on the triangle is the origin of the non-orthogonal coordinate system.
- The vectors from **a** to **b** and from **a** to **c** are taken as basis vectors.

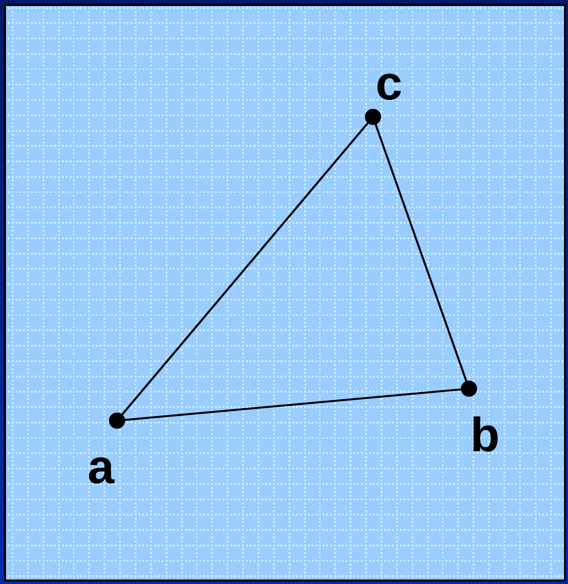


What are barycentric coordinates?

- We can express any point **p** coplanar to the triangle as:
  $\mathbf{p} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$
- Typically, we rewrite this as:
  $\mathbf{p}(\alpha,\beta,\gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$
  where $\alpha \equiv 1 - \beta - \gamma$
- $\mathbf{a} = \mathbf{p}(1,0,0)$, $\mathbf{b} = \mathbf{p}(0,1,0)$, $\mathbf{c} = \mathbf{p}(0,0,1)$



What are barycentric coordinates?



Some cool properties:

- Point **p** is inside the triangle if and only if

  $0 < \alpha < 1,$

  $0 < \beta < 1,$

  $0 < \gamma < 1$

- If one component is zero, **p** is on an edge.
- If two components are zero, **p** is on a vertex.
- The coordinates can be used as weighting factors for properties of the vertices, like color.

Barycentric coordinates

Chalkboard examples:

- Conversion from 2D Cartesian
- Conversion from 3D Cartesian