Announcements

Office hours for Jernej Barbic have changed:

New office hours:

Tuesday (same as before)
1 p.m. – 2 p.m. in the cluster

Thursday (changed)
1:45 p.m. - 2:45 p.m. in the cluster

Announcements (Contd.)

A link (URL) to the *libpicio* library has been added to the course website.

Libpicio library:

- read and write TIFF, XPM, JPEG formats
- used in assignment #1

Parametric Curves

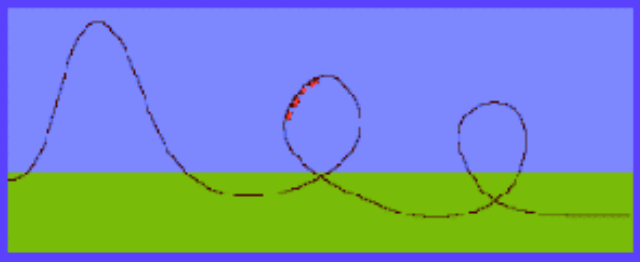
Modeling
Parametric Curves (Splines)

Chapter 3 in Watt

September 17 2002

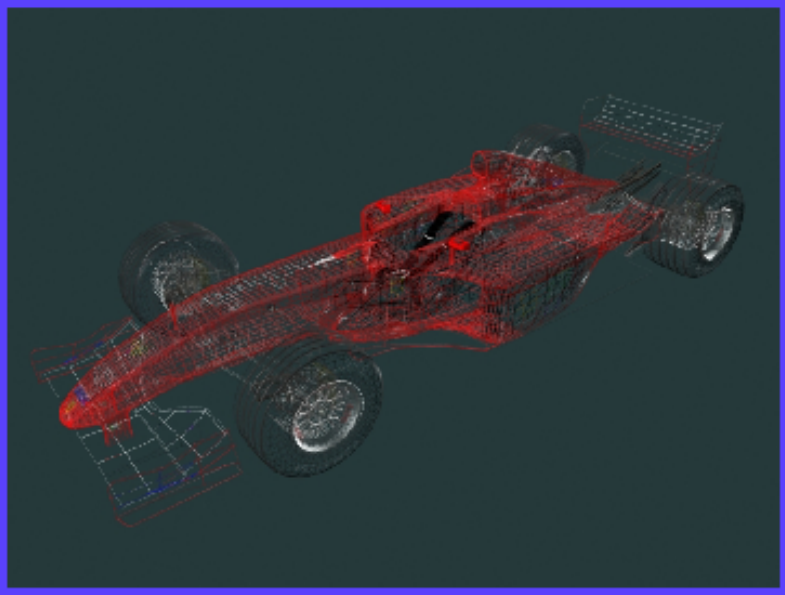
Roller coaster

- Next programming assignment involves creating a 3D roller coaster animation
- We must model the 3D curve describing the roller coaster, but how?
- How to make the simulation obey the laws of gravity?



Modeling Complex Shapes

- We want to build models of very complicated objects
- An equation for a sphere is possible, but how about an equation for a telephone, or a face, or a cloud?
- Complexity is achieved using simple pieces
  - polygons, parametric curves and surfaces, or implicit curves and surfaces
  - This lecture: parametric curves



What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering

Curve Representations

- **Explicit: y = f(x)**

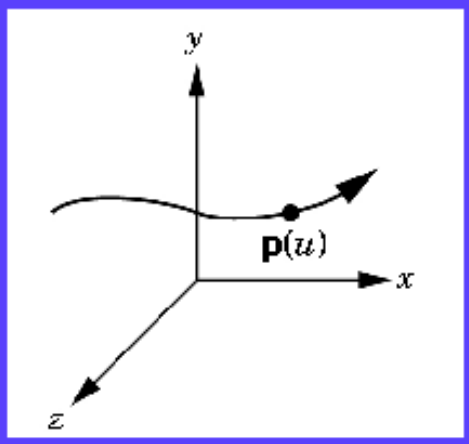
$$y = mx + b \qquad y = x^2$$

  - **Must be a function (single-valued):**
  - **Big limitation—vertical lines?**

- **Parametric: (x,y) = (f(u),g(u))**

$$(x, y) = (\cos u, \sin u)$$

  - + **Easy to specify, modify, control**
  - **Extra "hidden" variable u, the *parameter***



- **Implicit: f(x,y) = 0**

$$x^2 + y^2 - r^2 = 0$$

  - + **Y can be multiple valued function of x**
  - **Hard to specify, modify, control**

Parameterization of a Curve

- *Parameterization* of a curve: how a change in u moves you along a given curve in xyz space.
- There are an infinite number of parameterizations of a given curve. Slow, fast, speed continuous or discontinuous, clockwise (CW) or CCW…

Polynomial Interpolation

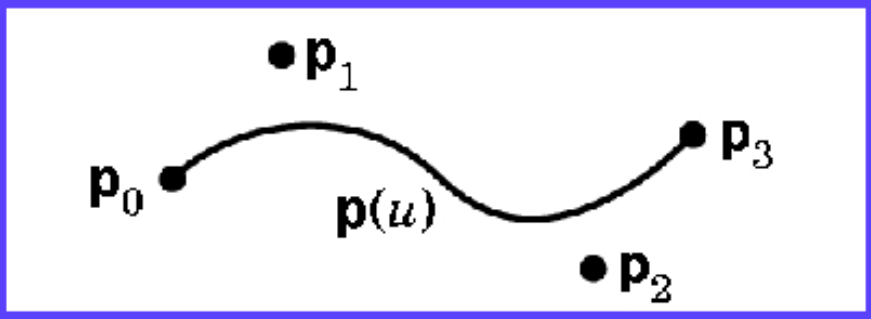
- An *n*-th degree polynomial fits a curve to n+1 points
  - called Lagrange Interpolation
  - result is a curve that is too wiggly, change to any control point affects entire curve (nonlocal) – *this method is poor*
- We usually want the curve to be as smooth as possible
  - minimize the wiggles
  - high-degree polynomials are bad

Splines: Piecewise Polynomials

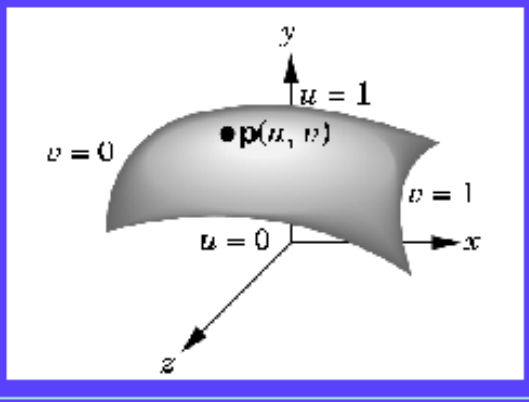
- A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points
- *Cubic piecewise* polynomials are the most common:
  - piecewise definiton gives local control
  - lowest order polynomials that interpolate two points and allow the gradient at each point to be defined - C^1 continuity is possible
  - Higher or lower degrees are possible, of course

Splines (contd.)

- Types of splines:
  - Hermite Splines
  - Bezier Splines
  - Catmull-Rom Splines
  - Natural Cubic Splines
  - B-Splines
  - NURBS



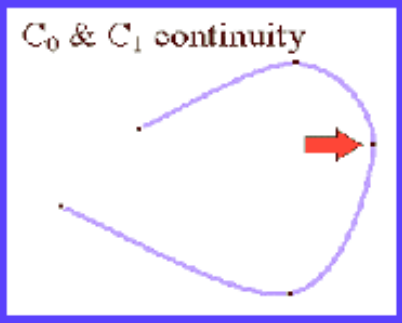
- Splines can be used to model surfaces



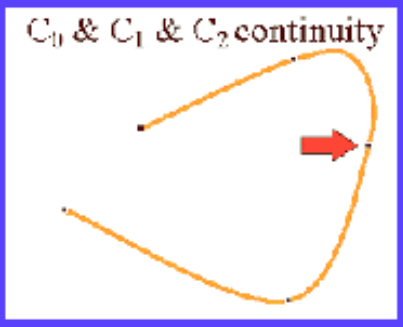
Piecewise Polynomials

- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely

Continuous in position



Continuous in position and tangent vector



Continuous in position, tangent, and curvature

Cubic Curves in 3D

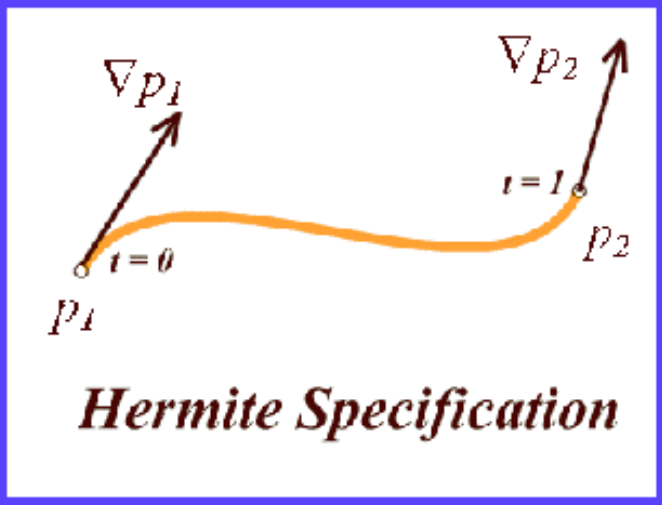
- **Cubic polynomial**
  - $x(u) = au^3+bu^2+cu+d$ = **ua**
  - **where u** = $[u^3 \;\; u^2 \;\; u \;\; 1]$, **a** = $[a \;\; b \;\; c \;\; d]^T$
- **Three cubic polynomials, one for each coordinate**
  - $x(u) = a_x u^3+b_x u^2+c_x u+d_x$
  - $y(u) = a_y u^3+b_y u^2+c_y u+d_y$
  - $z(u) = a_z u^3+b_z u^2+c_z u+d_z$
- **In matrix notation**

$$[x(u) \quad y(u) \quad z(u)] = [u^3 \quad u^2 \quad u \quad 1] \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

- **Or simply** **x = uA**

An Illustrative Example

- Cubic Hermite Splines



That is, we want a way to specify the end points and the slope at the end points!

Deriving Hermite Splines

- **Four constraints:** value and slope (or in 3-D, position and tangent vector) at beginning and end of interval [0,1]

$x(0) = x_1$

$x(1) = x_2$

$x'(0) = x_1'$

$x'(1) = x_2'$

primes on left side denote derivative; primes on right denote slope constants

- **Assume cubic form:** $x(u) = au^3+bu^2+cu+d$
- **Four unknowns:** a, b, c, d

The Cubic Hermite Spline Equation

- Using some algebra, we obtain:

$$[x \;\; y \;\; z] = [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \frac{dx_1}{du} & \frac{dy_1}{du} & \frac{dz_1}{du} \\ \frac{dx_2}{du} & \frac{dy_2}{du} & \frac{dz_2}{du} \end{bmatrix}$$

point that gets drawn

basis

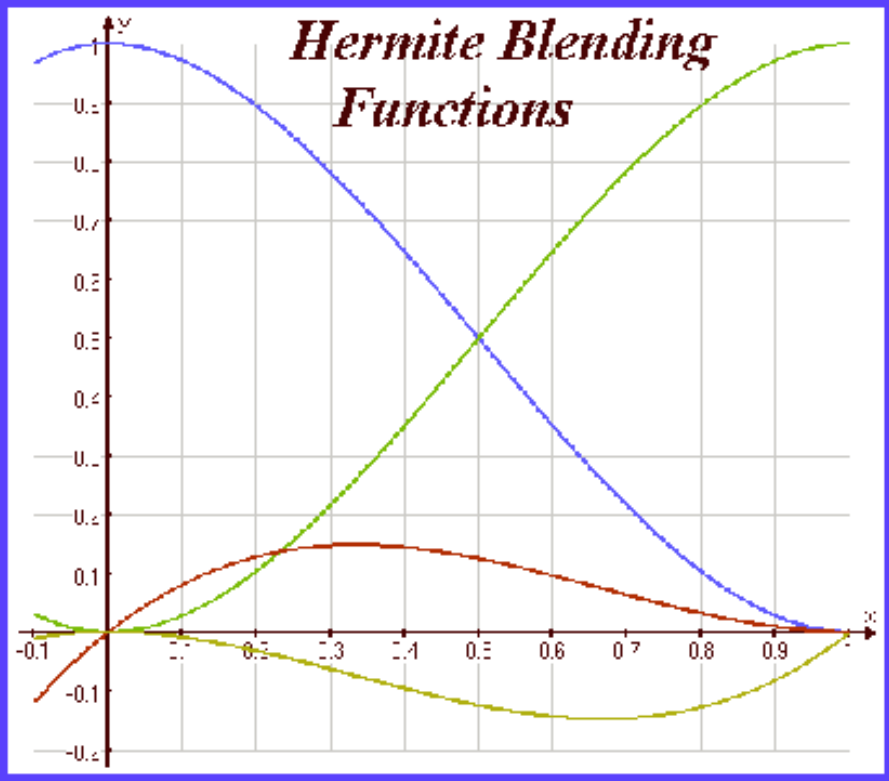
control matrix (what the user gets to pick)

- This form typical for splines
  - basis matrix and meaning of control matrix change with the spline type

Four Basis Functions for Hermite splines

$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

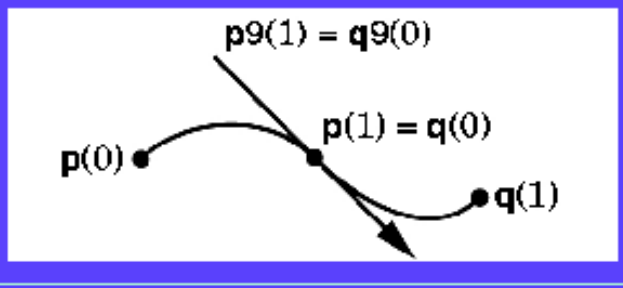
4 Basis Functions



Every cubic Hermite spline is a linear combination (blend) of these 4 functions

Piecing together Hermite Curves

- It's easy to make a multi-segment Hermite spline
  - each piece is specified by a cubic Hermite curve
  - just specify the position and tangent at each "joint"
  - the pieces fit together with matched positions and first derivatives
  - gives C1 continuity
- Given a list of points & tangents, you can build a piecewise cubic that passes through all the points by calculating the Hermite cubic for each segment
- The points that the curve has to pass through are called *knots* or *knot points*



Bezier Curves*

- Another variant of the same game
- Instead of endpoints and tangents, four control points
  - points P1 and P4 are on the curve
  - points P2 and P3 are off the curve
  - X(0) = P1, X(1) = P4,
  - X'(0) = 3(P2-P1), X'(1) = 3(P4 - P3)

- Variant of the Hermite spline
  - basis matrix derived from the Hermite basis (or from scratch)
- Convex Hull property
  - curve contained within convex hull of control points
- Gives more uniform control knobs (series of points) than Hermite
- Scale factor (3) is chosen to make "velocity" approximately constant

The Bezier Spline Matrix*

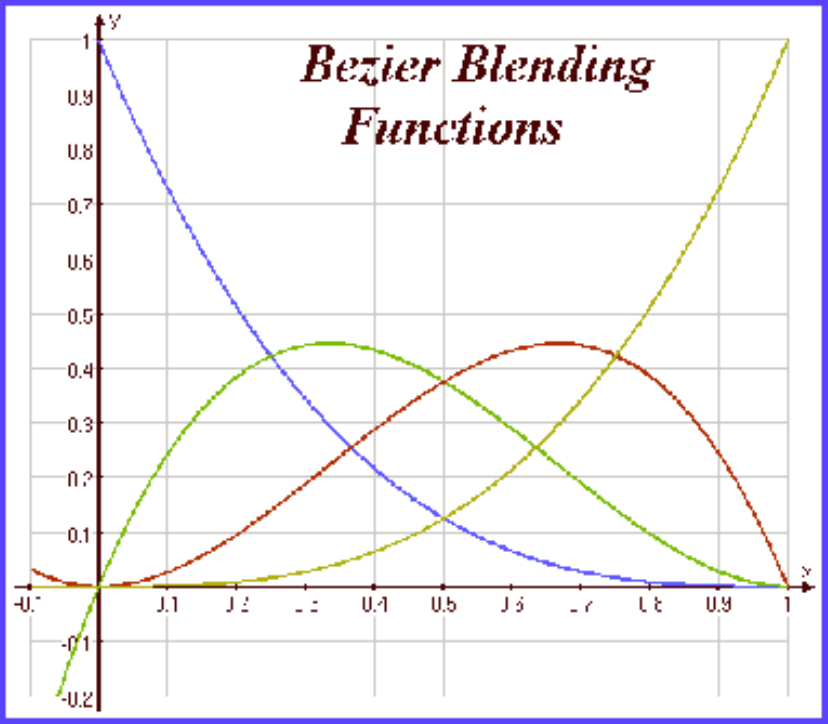
$$[x \;\; y \;\; z] = [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

Hermite basis **Bezier to Hermite** **Bezier control vector**

$$= [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

Bezier basis **Bezier control vector**

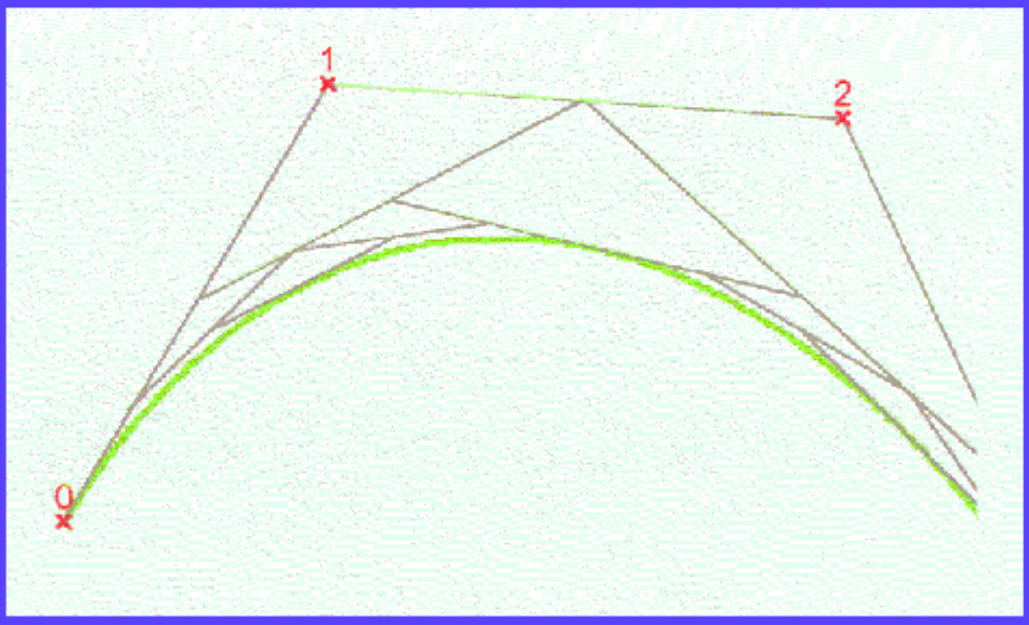
Bezier Blending Functions*



$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

- Also known as the order 4, degree 3 Bernstein polynomials
- Nonnegative, sum to 1
- The entire curve lies inside the polyhedron bounded by the control points

It's easy to subdivide Bezier curves*



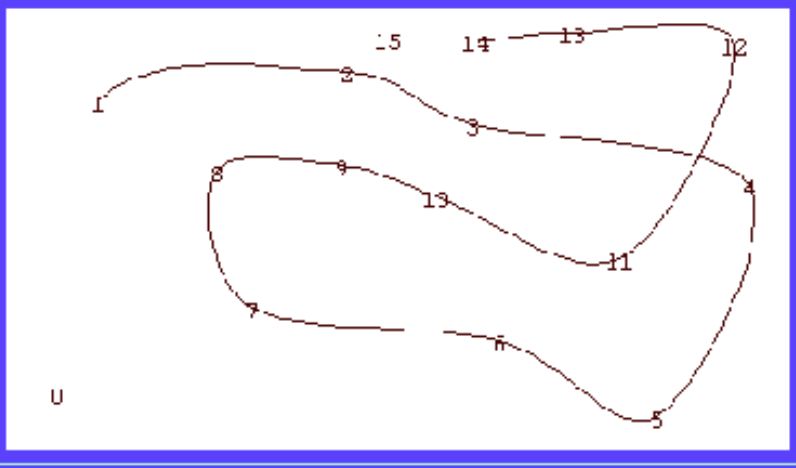
Each half is a Bezier curve, therefore it is easy to draw them by subdivision

Catmull-Rom Splines

- Use for the roller-coaster (next programming assignment)
- With Hermite splines, the designer must arrange for consecutive tangents to be collinear, to get C^1 continuity. Similar for Bezier. This gets tedious.
- Catmull-Rom: an interpolating cubic spline with *built-in C^1 continuity*.
- Compared to Hermite/Bezier: fewer control points required, but less freedom.

Catmull-Rom Splines, contd.

- Given n control points in 3-D: p_1, p_2, …, p_n,
  - Tangent at p_i given by $s*(p_{i+1} - p_{i-1})$ for i=2..n-1, for some s
  - s is tension parameter: determines the magnitude (but not direction!) of the required tangent at point p_i
  - What about endpoint tangents? (several good answers: extrapolate, or use extra control points p_0, p_{n+1})
  - Now we have positions and tangents at each knot – a Hermite specification.
  - Curve between p_i and p_{i+1} is determined by p_{i-1}, p_i, p_{i+1}, p_{i+2}



Catmull-Rom Spline Matrix

$$[x \;\; y \;\; z] = [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

spline coefficients **CR basis** **control vector**

- **Derived similarly to Hermite and Bezier**
- **Parameter s is typically set to s=1/2.**

Splines with More Continuity?

- So far, only C^1 continuity.
- How could we get C^2 continuity at control points?

- Possible answers:
  - Use higher degree polynomials

    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  - Give up local control → natural cubic splines

    A change to any control point affects the entire curve
  - Give up interpolation → cubic B-splines

    Curve goes near, but not through, the control points

Comparison of Basic Cubic Splines

Type	Local Control	Continuity	Interpolation
Hermite	YES	C1	YES
Bezier	YES	C1	YES
Catmull-Rom	YES	C1	YES
Natural	NO	C2	YES
B-Splines	YES	C2	NO

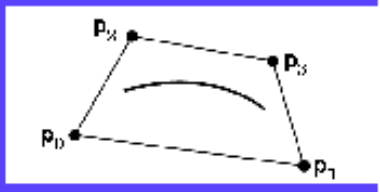
- Summary
  - Can't get C2, interpolation and local control with cubics - very sad.

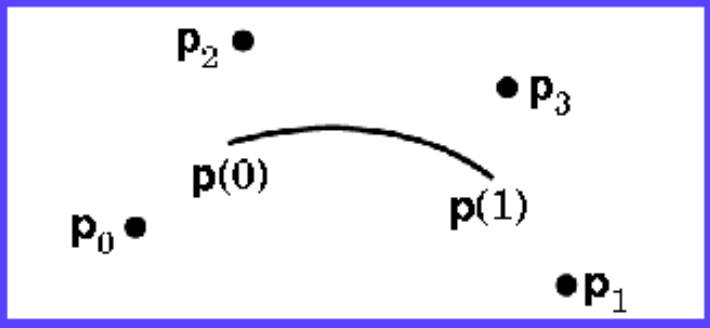
Natural Cubic Splines*

- If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines*
- It's a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)
- Finding all the right weights is a *global* calculation (solve tridiagonal linear system)

B-Splines*

- Give up interpolation
  - the curve passes *near* the control points
  - best generated with interactive placement (because it's hard to guess where the curve will go)
- Curve obeys the convex hull property
- C2 continuity and local control are good compensation for loss of interpolation



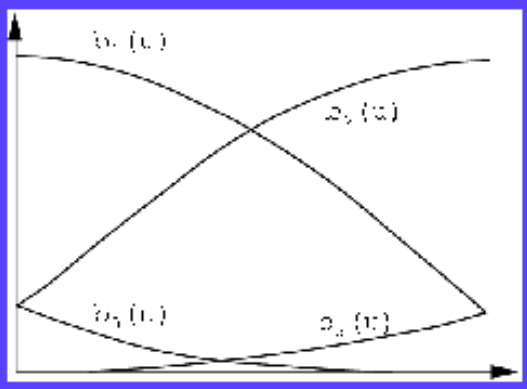


B-Spline Basis*

- We always need 3 more control points than spline pieces

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{BSi} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



Other common types of splines*

- Non-uniform Splines
- Non-Uniform Rational Cubic curves (NURBS)

How to Draw Spline Curves

- Basis matrix eqn. allows same code to draw any spline type
- Method 1: brute force
  - Calculate the coefficients
  - For each cubic segment, vary *u* from *0* to *1* (fixed step size)
  - Plug in *u* value, matrix multiply to compute position on curve
  - Draw line segment from last position to current position
- What's wrong with this approach?
  - Draws in even steps of u
  - Even steps of u ≠ even steps of x
  - Line length will vary over the curve
  - Want to bound line length
    - » too long: curve looks jagged
    - » too short: curve is slow to draw

Drawing Splines, 2

- **Method 2: recursive subdivision - vary step size to draw short lines**

```
Subdivide(u0,u1,maxlinelength)
    umid = (u0 + u1)/2
    x0 = F(u0)
    x1 = F(u1)
    if |x1 - x0| > maxlinelength
       Subdivide(u0,umid,maxlinelength)
       Subdivide(umid,u1,maxlinelength)
    else drawline(x0,x1)
```

- **Variant on Method 2 - subdivide based on curvature**
  - **replace condition in "if" statement with straightness criterion**
  - **draws fewer lines in flatter regions of the curve**

In Summary...

- Summary:
  - piecewise cubic is generally sufficient
  - define conditions on the curves and their continuity

- Things to know:
  - basic curve properties (what are the conditions, controls, and properties for each spline type)
  - generic matrix formula for uniform cubic splines $x(u) = uBG$
  - given definition derive a basis matrix (do not memorize matrices themselves)