Types, Abstraction, and Parametric Polymorphism — for Domains

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The Syntax of Types and Contexts

Let T be a finite set of *type variables.* Then Θ_T, the set of *types* over T, is the least family of sets such that

- If $\tau \in T$, then $\tau \in \Theta_T$.
- If $\theta, \theta' \in \Theta_T$, then $\theta \to \theta' \in \Theta_T$.
- If $\theta \in \Theta_{T \cup \{\tau\}}$, then $\forall \tau.\, \theta \in \Theta_T$.

Also, Π_T is the set of *contexts* over T, which are functions

$$\pi = [\, v_1 \colon \theta_1 \mid \ldots \mid v_n \colon \theta_n \,]$$

from a finite set $\{v_1, \ldots, v_n\}$ of *ordinary variables* to Θ_T.

The Semantics of Types and Contexts

We assume that **Dom** is the set of *domains*, which are certain complete partial orders with least elements.

Suppose T is a finite set of type variables, and

$$D \in \mathbf{Dom}^T$$

is a *domain assignment*. Then

$$D^\# \in \mathbf{Dom}^{(\Theta_T \cup \Pi_T)}$$

is the extension of D to types and type assignments over T, such that

$$\begin{aligned} D^\#(\tau) &= D(\tau) \\ D^\#(\theta \to \theta') &= D^\#(\theta) \Rightarrow D^\#(\theta') \\ D^\#(\forall\tau.\ \theta) &\subset \textstyle\prod_{d \in \mathbf{Dom}} [\, D \mid \tau{:}\, d\,]^\#(\theta) \\ D^\#(\pi) &= \textstyle\prod_{v \in \mathrm{dom}\,\pi} D^\#(\pi(v)), \end{aligned}$$

where

- $d \Rightarrow d'$ denotes the domain of continuous functions from d to d' (ordered pointwise),
- $\prod_{d \in S}$ denotes a Cartesian product indexed by S (ordered componentwise).

Polymorphic Extensionality

Since $D^{\#}(\forall\tau.\ \theta)$ is a subset of

$$\textstyle\prod_{d\in\mathbf{Dom}}[\, D \mid \tau\colon d\,]^{\#}(\theta) = \\ \{\, p \mid \mathrm{dom}\, p = \mathbf{Dom} \text{ and } \forall d \in \mathbf{Dom}.\ p[d] \in [\, D \mid \tau\colon d\,]^{\#}(\theta)\,\},$$

polymorphic functions are *extensional*, i.e., for all $p, p' \in D^{\#}(\forall\tau.\ \theta)$,

$$(\forall d \in \mathbf{Dom}.\ p[d] = p'[d]) \text{ implies } p = p'.$$

Polymorphic Application

We also assume that polymorphic application is strict, i.e.,

$$\perp[d] = \perp.$$

Complications we are Ignoring

- The meanings of types may be entities (such as retractions of a universal domain) that denote domains.
- Products and Lifting
- Coproducts and Predomains

The Syntax of Terms

$$\frac{\pi \in \Pi_T \qquad v \in \operatorname{dom}\pi}{\pi \vdash_T v : \pi(v)}$$

$$\frac{\pi \vdash_T e_1 : \theta \rightarrow \theta' \qquad \pi \vdash_T e_2 : \theta}{\pi \vdash_T e_1 e_2 : \theta'}$$

$$\frac{[\, \pi \mid v{:}\, \theta \,] \vdash_T e : \theta'}{\pi \vdash_T \lambda v{:}\, \theta.\ e : \theta \rightarrow \theta'}$$

$$\frac{\pi \vdash_T e : \forall\tau.\ \theta \qquad \theta' \in \Theta_T}{\pi \vdash_T e[\theta'] : (\theta/\tau \rightarrow \theta')}$$

$$\frac{\pi \vdash_{T \cup \{\tau\}} e : \theta \qquad \pi \in \Pi_T}{\pi \vdash_T \Lambda\tau.\ e : \forall\tau.\ \theta}$$

$$\frac{\pi \vdash_T e : \theta \rightarrow \theta}{\pi \vdash_T \mathbf{Y}_\theta e : \theta}$$

The Semantics of Terms

When $\pi \vdash_T e : \theta$ and $D \in \mathbf{Dom}^T$,

$$[\![e]\!]_T^{\pi,\theta} D \in D^{\#}(\pi) \Rightarrow D^{\#}(\theta)$$

is defined by

$$\frac{\pi \in \Pi_T \qquad v \in \operatorname{dom} \pi}{\pi \vdash_T v : \pi(v)} \qquad [\![v]\!]_T^{\pi,\pi(v)} D\eta = \eta(v)$$

$$\frac{\pi \vdash_T e_1 : \theta \to \theta' \qquad \pi \vdash_T e_2 : \theta}{\pi \vdash_T e_1 e_2 : \theta'}$$

$$[\![e_1 e_2]\!]_T^{\pi,\theta'} D\eta = ([\![e_1]\!]_T^{\pi,\theta \to \theta'} D\eta)([\![e_2]\!]_T^{\pi,\theta} D\eta)$$

$$\frac{[\, \pi \mid v{:}\, \theta \,] \vdash_T e : \theta'}{\pi \vdash_T \lambda v{:}\, \theta.\ e : \theta \to \theta'} \qquad [\![\lambda v{:}\, \theta.\ e]\!]_T^{\pi,\theta \to \theta'} D\eta = \lambda x \in D^{\#}(\theta).\ [\![e]\!]_T^{[\pi \mid v:\theta],\theta'} D[\, \eta \mid v{:}\, x \,]$$

$$\frac{\pi \vdash_T e : \forall \tau.\ \theta \qquad \theta' \in \Theta_T}{\pi \vdash_T e[\theta'] : (\theta/\tau \to \theta')} \qquad [\![e[\theta']]\!]_T^{\pi,(\theta/\tau \to \theta')} D\eta = [\![e]\!]_T^{\pi,\forall \tau.\ \theta} D\eta[D^{\#}(\theta')]$$

$$\frac{\pi \vdash_{T \cup \{\tau\}} e : \theta \qquad \pi \in \Pi_T}{\pi \vdash_T \Lambda \tau.\ e : \forall \tau.\ \theta} \qquad [\![\Lambda \tau.\ e]\!]_T^{\pi,\forall \tau.\ \theta} D\eta = \lambda d \in \mathbf{Dom}.\ [\![e]\!]_{T \cup \{\tau\}}^{\pi,\theta} [\, D \mid \tau{:}\, d \,]\eta$$

$$\frac{\pi \vdash_T e : \theta \to \theta}{\pi \vdash_T \mathbf{Y}_\theta e : \theta} \qquad [\![\mathbf{Y}_\theta e]\!]_T^{\pi,\theta} D\eta = Y_{D^{\#}(\theta)} [\![e]\!]_T^{\pi,\theta \to \theta} D\eta$$

More Simply

$$\frac{\pi \in \Pi_T \qquad v \in \operatorname{dom}\pi}{\pi \vdash_T v : \pi(v)} \qquad [\![v]\!]D\eta = \eta(v)$$

$$\frac{\pi \vdash_T e_1 : \theta \to \theta' \qquad \pi \vdash_T e_2 : \theta}{\pi \vdash_T e_1 e_2 : \theta'}$$

$$[\![e_1 e_2]\!]D\eta = ([\![e_1]\!]D\eta)([\![e_2]\!]D\eta)$$

$$\frac{[\,\pi \mid v\colon\theta\,] \vdash_T e : \theta'}{\pi \vdash_T \lambda v\colon\theta.\ e : \theta \to \theta'} \qquad \begin{array}{l} [\![\lambda v\colon\theta.\ e]\!]D\eta = \\ \qquad \lambda x \in D^{\#}(\theta).\ [\![e]\!]D[\,\eta \mid v\colon x\,] \end{array}$$

$$\frac{\pi \vdash_T e : \forall\tau.\ \theta \qquad \theta' \in \Theta_T}{\pi \vdash_T e[\theta'] : (\theta/\tau \to \theta')} \qquad [\![e[\theta']]\!]D\eta = [\![e]\!]D\eta[D^{\#}(\theta')]$$

$$\frac{\pi \vdash_{T\cup\{\tau\}} e : \theta \qquad \pi \in \Pi_T}{\pi \vdash_T \Lambda\tau.\ e : \forall\tau.\ \theta} \qquad \begin{array}{l} [\![\Lambda\tau.\ e]\!]D\eta = \\ \qquad \lambda d \in \mathbf{Dom}.\ [\![e]\!][\,D \mid \tau\colon d\,]\eta \end{array}$$

$$\frac{\pi \vdash_T e : \theta \to \theta}{\pi \vdash_T \mathbf{Y}_\theta e : \theta} \qquad [\![\mathbf{Y}_\theta e]\!]D\eta = Y_{D^{\#}(\theta)}[\![e]\!]D\eta$$

Complete Relations

A binary relation r between domains d_1 and d_2 is said to be *complete* iff

- $\langle \bot, \bot \rangle \in r$.
- If
$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$$
$$y_0 \sqsubseteq y_1 \sqsubseteq y_2 \sqsubseteq \cdots$$
$$\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in r,$$
then
$$\langle \textstyle\bigsqcup_{i=0}^{\infty} x_i, \bigsqcup_{i=0}^{\infty} y_i \rangle \in r.$$

The Relational Semantics of Types and Contexts

Suppose T is a finite set of type variables, $D_1, D_2 \in \mathbf{Dom}^T$, and R is a function from T to relations such that, for all $\tau \in T$,

$R(\tau)$ is a complete relation between $D_1(\tau)$ and $D_2(\tau)$.

Then R (called a *relation assignment* between D_1 and D_2) can be extended to a function $R^{\#}$ on types and type assignments such that, for all $\tau \in \Theta_T$ and $\pi \in \Pi_T$,

$R^{\#}(\tau)$ is a complete relation between $D_1^{\#}(\tau)$ and $D_2^{\#}(\tau)$,

$R^{\#}(\pi)$ is a complete relation between $D_1^{\#}(\pi)$ and $D_2^{\#}(\pi)$,

by the following definitions:

- $R^{\#}(\tau) = R(\tau)$.
- $R^{\#}(\theta \to \theta') = R^{\#}(\theta) \overset{r}{\Longrightarrow} R^{\#}(\theta')$, where

  $\langle f_1, f_2 \rangle \in r \overset{r}{\Longrightarrow} r'$ iff

  $$\text{for all } \langle x_1, x_2 \rangle \in r,\ \langle f_1(x_1), f_2(x_2) \rangle \in r'.$$

- $\langle p_1, p_2 \rangle \in R^{\#}(\forall \tau.\ \theta)$ iff, for all domains d_1 and d_2, and all complete relations r between d_1 and d_2,

  $$\langle p_1[d_1], p_2[d_2] \rangle \in [\, R \mid \tau\colon r \,]^{\#}(\theta).$$

- $\langle \eta_1, \eta_2 \rangle \in R^{\#}(\pi)$ iff, for all $v \in \operatorname{dom} \pi$,

  $$\langle \eta_1(v), \eta_2(v) \rangle \in R^{\#}(\pi(v)).$$

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The Logical Relations (Abstraction) Theorem

If

- $D_1, D_2 \in \mathbf{Dom}^T$,
- R is a relation assignment between D_1 and D_2,
- $\pi \vdash_T e : \theta$,
- $\langle \eta_1, \eta_2 \rangle \in R^{\#}\pi$,

then

$$\langle [\![e]\!]_T^{\pi,\theta} D_1 \eta_1, [\![e]\!]_T^{\pi,\theta} D_2 \eta_2 \rangle \in R^{\#}\theta.$$

Parametricity

When $p \in D^{\#}(\forall\tau.\ \theta)$ for some $D \in \mathbf{Dom}^T$, we say that p is *parametric* iff for all domains d_1 and d_2, and all complete relations r between d_1 and d_2,

$$\langle p[d_1], p[d_2]\rangle \in [\, I_D \mid \tau\colon r\,]^{\#}(\theta),$$

where I_D is the relation assignment that assigns the identity relation on $D(\tau)$ to each $\tau \in T$.

Then we restrict the meaning of polymorphic types to include only parameteric polymorphic functions:

$$D^{\#}(\forall\tau.\ \theta) \subseteq \{\, p \in \textstyle\prod_{d\in\mathbf{Dom}}[\, D \mid \tau\colon d\,]^{\#}(\theta) \mid p \text{ is parametric}\,\},$$

to obtain:

The Identity Extension Lemma

For all $\theta \in \Theta_T$, $\pi \in \Pi_T$, and $D \in \mathbf{Dom}^T$,

$$I_D^{\#}(\theta) \text{ is the identity relation on } D^{\#}(\theta),$$

$$I_D^{\#}(\pi) \text{ is the identity relation on } D^{\#}(\pi).$$

Why Parametricity is Permissible

Suppose $\pi \vdash_T e : \forall\tau.\ \theta$, $D \in \mathbf{Dom}^T$, and $\eta \in D^\#(\pi)$. By the identity extension lemma,

$$\langle \eta, \eta \rangle \in I_D^\#(\pi).$$

Then the logical relations theorem gives

$$\langle [\![e]\!]D\eta, [\![e]\!]D\eta \rangle \in I_D^\#(\forall\tau.\ \theta).$$

Thus, by the definition of $R^\#(\forall\tau.\ \theta)$, for all domains d_1 and d_2, and all complete relations r between s_1 and d_2,

$$\langle ([\![e]\!]D\eta)[d_1], ([\![e]\!]D\eta)[d_2] \rangle \in [\, I_D \mid \tau\colon r \,]^\#(\theta),$$

i.e., $[\![e]\!]D\eta$ is parametric.

More Complications we are Ignoring

- Multinary relations
- Kripke relations

m$\mathcal{T}$-Algebras

A (continuous) $m\mathcal{T}$-algebra is one in which

- The carrier is a domain,
- The operations are continuous functions.
- Homomorphisms are strict continuous functions.
- The arities are endofunctors on the category of domains.

A *signature* is a finite list $\mathcal{T}_1, \ldots \mathcal{T}_k$, of (covariant) endofunctors on the category of domains. Each $\mathcal{T}_i$ is called the *arity of the operator* σ_i. When $\mathcal{T}_i(x) = x^n$, σ_i is said to be *n-ary*.

A m$\mathcal{T}$-algebra A consists of:

- A domain $|A|$ called the *carrier*,
- For each operator σ_i, an *operation* σ_i^A, which is a continuous function from $\mathcal{T}_i|A|$ to $|A|$.

When A and B are m$\mathcal{T}$-algebras with the same signature, a *homomorphism from A to B* is a strict continuous function ψ from $|A|$ to $|B|$ such that, for each σ_i,

$$\begin{tikzcd}
\mathcal{T}_i|A| \arrow[r, "\mathcal{T}_i\psi"] \arrow[d, "\sigma_i^A"] & \mathcal{T}_i|B| \arrow[d, "\sigma_i^B"] \\
|A| \arrow[r, "\psi"] & |B|
\end{tikzcd}$$

Expressible And Relationally Consistent Functors

A functor $\mathcal{T}$ is *expressible* iff

- There is a type $\mathsf{T}(\tau) \in \Theta_{\{\tau\}}$ such that

$$[\,\tau\colon d\,]^{\#}(\mathsf{T}(\tau)) = \mathcal{T}d.$$

- There is a term $\mathsf{T}^{\tau,\tau'}(v)$ satisfying

$$[\,v\colon \tau \to \tau'\,] \vdash_{\{\tau,\tau'\}} \mathsf{T}^{\tau,\tau'}(v) : \mathsf{T}(\tau) \to \mathsf{T}(\tau'),$$

  such that, when $f \in d \Rightarrow d'$,

$$[\![\mathsf{T}^{\tau,\tau'}(v)]\!][\,\tau\colon d \mid \tau'\colon d'\,][\,v\colon f\,] = \mathcal{T}f.$$

An expressible functor $\mathcal{T}$ is *relationally consistent* iff, when $\psi \in d \to d'$ is viewed as a relation,

$$[\,\tau\colon \psi\,]^{\#}(\mathsf{T}(\tau)) = \mathcal{T}\psi.$$

Constructing a Weakly Initial m$\mathcal{T}$-Algebra

Given a signature $\mathcal{T}_1, \ldots, \mathcal{T}_k$, let

$$\mathsf{P} \stackrel{\text{def}}{=} \forall\tau.\ (\mathsf{T}_1(\tau) \to \tau) \to \cdots \to (\mathsf{T}_k(\tau) \to \tau) \to \tau$$
$$\text{satisfying } \mathsf{P} \in \Theta_{\{\}}$$

$$\mathsf{M} \stackrel{\text{def}}{=} \lambda\mathsf{p}{:}\,\mathsf{P}.\ \mathsf{p}[\tau]\mathsf{f}_1 \cdots \mathsf{f}_k$$
$$\text{satisfying } [\,\mathsf{f}_1{:}\,\mathsf{T}_1(\tau) \to \tau \mid \ldots \mid \mathsf{f}_k{:}\,\mathsf{T}_k(\tau) \to \tau\,] \vdash_{\{\tau\}} \mathsf{M} : \mathsf{P} \to \tau$$

$$\mathsf{H}_i \stackrel{\text{def}}{=} \lambda\mathsf{q}{:}\,\mathsf{T}_i(\mathsf{P}).\ \Lambda\tau.\ \lambda\mathsf{f}_1{:}\,\mathsf{T}_1(\tau) \to \tau.\ \cdots \lambda\mathsf{f}_k{:}\,\mathsf{T}_k(\tau) \to \tau.\ \mathsf{f}_i(\mathsf{T}_i^{\mathsf{P},\tau}(\mathsf{M})\mathsf{q})$$
$$\text{satisfying } [\,] \vdash_{\{\}} \mathsf{H}_i : \mathsf{T}_i(\mathsf{P}) \to \mathsf{P}.$$

Then, for each operator, by reduction:

$$\begin{aligned}
\mathsf{M}(\mathsf{H}_i(\mathsf{q})) &\leadsto (\lambda\mathsf{p}.\ \mathsf{p}[\tau]\mathsf{f}_1 \cdots \mathsf{f}_k)(\Lambda\tau.\ \lambda\mathsf{f}_1.\ \cdots \lambda\mathsf{f}_k.\ \mathsf{f}_i(\mathsf{T}_i^{\mathsf{P},\tau}(\mathsf{M})\mathsf{q})) \\
&\leadsto (\Lambda\tau.\ \lambda\mathsf{f}_1.\ \cdots \lambda\mathsf{f}_k.\ \mathsf{f}_i(\mathsf{T}_i^{\mathsf{P},\tau}(\mathsf{M})\mathsf{q}))[\tau]\mathsf{f}_1 \cdots \mathsf{f}_k \\
&\leadsto^* \mathsf{f}_i(\mathsf{T}_i^{\mathsf{P},\tau}(\mathsf{M})\mathsf{q}),
\end{aligned}$$

i.e.,

$$\begin{tikzcd}
\mathsf{T}_i(\mathsf{P}) \arrow[r, "\mathsf{T}_i^{\mathsf{P},\tau}(\mathsf{M})"] \arrow[d, "\mathsf{H}_i"] & \mathsf{T}_i(\tau) \arrow[d, "\mathsf{f}_i"] \\
\mathsf{P} \arrow[r, "\mathsf{M}"] & \tau
\end{tikzcd}$$

commutes syntactically.

Constructing a Weakly Initial m$\mathcal{T}$-Algebra (continued)

Thus, we may define the m$\mathcal{T}$-algebra N such that

$$\begin{aligned} |N| &= [\,]^{\#}\mathsf{P} \\ \sigma_i^N &= [\![\mathsf{H}_i]\!][\,][\,] \in \mathcal{T}_i|N| \Rightarrow |N|. \end{aligned}$$

Moreover, for any m$\mathcal{T}$-algebra A with the same signature, we may define the function

$$\phi_A = [\![\mathsf{M}]\!][\,\tau\colon |A|\,][\,\mathsf{f}_1\colon \sigma_1^A \mid \ldots \mid \mathsf{f}_k\colon \sigma_k^A\,]$$

which satisfies

$$\begin{aligned} \phi_A(\sigma_i^N) &= [\![\mathsf{M}(\mathsf{H}_i(\mathsf{q}))]\!][\,\tau\colon |A|\,][\mathsf{f}_1\colon \sigma_1^A \mid \cdots \mid \mathsf{f}_k\colon \sigma_k^A \mid \mathsf{q}\colon q] \\ &= [\![\mathsf{f}_i(\mathsf{T}_i^{\mathsf{P},\tau}(\mathsf{M})\mathsf{q})]\!][\,\tau\colon |A|\,][\mathsf{f}_1\colon \sigma_1^A \mid \cdots \mid \mathsf{f}_k\colon \sigma_k^A \mid \mathsf{q}\colon q] \\ &= \sigma_i(\mathcal{T}_i(\phi_A)(q)), \end{aligned}$$

i.e.,

$$\begin{tikzcd} \mathcal{T}_i|N| \arrow[r, "\mathcal{T}_i\phi_A"] \arrow[d, "\sigma_i^N"] & \mathcal{T}_i|A| \arrow[d, "\sigma_i^A"] \\ |N| \arrow[r, "\phi_A"] & |A| \end{tikzcd}$$

Moreover, since polymorphic application is strict,

$$\begin{aligned} \phi_A(\perp) &= [\![\mathsf{M}(\mathsf{p})]\!][\,\tau\colon |A|\,][\mathsf{f}_1\colon \sigma_1^A \mid \cdots \mid \mathsf{f}_k\colon \sigma_k^A \mid \mathsf{p}\colon \perp] \\ &= [\![\mathsf{p}[\tau]\mathsf{f}_1 \cdots \mathsf{f}_k]\!][\,\tau\colon |A|\,][\mathsf{f}_1\colon \sigma_1^A \mid \cdots \mid \mathsf{f}_k\colon \sigma_k^A \mid \mathsf{p}\colon \perp] \\ &= \perp. \end{aligned}$$

Thus, ϕ_A is a homomorphism from N to A, and since A is any m$\mathcal{T}$-algebra with signature $\mathcal{T}_1, \ldots, \mathcal{T}_n$, N is weakly initial.

Strengthening the Weakly Initial Algebra

Henceforth, we assume the $\mathcal{T}_i$ are relationally consistent. Suppose ψ is a homomorphism from A to B. Since

$$\begin{tikzcd} \mathcal{T}_i|A| \arrow[r, "\mathcal{T}_i\psi"] \arrow[d, "\sigma_i^A"] & \mathcal{T}_i|B| \arrow[d, "\sigma_i^B"] \\ |A| \arrow[r, "\psi"] & |B| \end{tikzcd}$$

commutes, if we view ψ as a complete relation then

$$\forall \langle y_1, y_2 \rangle \in \mathcal{T}_i\psi.\ \langle \sigma_i^A y_1, \sigma_i^B y_2 \rangle \in \psi.$$

Then, since $\mathcal{T}$ is relationally consistent,

$$\forall \langle y_1, y_2 \rangle \in [\, \tau \colon \psi \,]^{\#}(\mathsf{T}_i(\tau)).\ \langle \sigma_i^A y_1, \sigma_i^B y_2 \rangle \in \psi,$$

so that

$$\langle \sigma_i^A, \sigma_i^B \rangle \in [\, \tau \colon \psi \,]^{\#}(\mathsf{T}_i(\tau) \to \tau).$$

When $p \in |N| = [\,]^{\#}(\forall\tau.\ (\mathsf{T}_1(\tau) \to \tau) \to \cdots \to (\mathsf{T}_k(\tau) \to \tau) \to \tau)$, parametricity gives

$$\langle p[|A|], p[|B|] \rangle \in [\, \tau \colon \psi \,]^{\#}((\mathsf{T}_1(\tau) \to \tau) \to \cdots \to (\mathsf{T}_k(\tau) \to \tau) \to \tau),$$

so that

$$\langle p[|A|]\sigma_1^A \cdots \sigma_k^A, p[|B|]\sigma_1^B \cdots \sigma_k^B \rangle \in [\, \tau \colon \psi \,]^{\#}(\tau) = \psi,$$

or, since ψ is a function,

$$\begin{tikzcd} & |N| \arrow[dl, "\phi_A = \lambda p.\ p[|A|]\sigma_1^A \cdots \sigma_k^A"'] \arrow[dr, "\phi_B = \lambda p.\ p[|B|]\sigma_1^B \cdots \sigma_k^B"] & \\ |A| \arrow[rr, "\psi"] & & |B| \end{tikzcd}$$

Strengthening the Algebra (continued)

As a special case, when ψ is a homomorphism from N to A:

$$\begin{tikzcd} & {|N|} \arrow[dl, "\phi_N"'] \arrow[dr, "\phi_A"] & \\ {|N|} \arrow[rr, "\psi"] & & {|A|} \end{tikzcd} \tag{1}$$

For any $d \in \mathbf{Dom}$ and $f_1 \in \mathcal{T}_1 d \Rightarrow d$, $\ldots$, $f_k \in \mathcal{T}_k d \Rightarrow d$, we can take A to be the algebra with carrier $|A| = d$ and operations $\sigma_i^A = f_i$. Then taking $\psi = \phi_A$ in (1), we have

$$\begin{tikzcd} & {|N|} \arrow[dl, "\phi_N"'] \arrow[dr, "\phi_A"] & \\ {|N|} \arrow[rr, "\phi_A"] & & d \end{tikzcd}$$

where $\phi_A = \lambda p.\ p[A]\sigma_1^A \cdots \sigma_k^A = \lambda p.\ p[d]f_1 \cdots f_k$. Thus

$$\phi_N(p)[d]f_1 \cdots f_k = p[d]f_1 \cdots f_k,$$

so by ordinary extensionality

$$\phi_N(p)[d] = p[d],$$

and by polymorphic extensionality

$$\phi_N(p) = p.$$

Then (1) gives $\psi = \phi_A$, so that N is an initial m$\mathcal{T}$-algebra.

The Separated Sum

We write

- $(d_1 + d_2)_\perp$ for the separated sum of d_1 and d_2,
- $\iota_1 \in d_1 \Rightarrow (d_1 + d_2)_\perp$ and $\iota_2 \in d_2 \Rightarrow (d_1 + d_2)_\perp$ for the associated injections,

Then, for any continuous functions $f_1 \in d_1 \Rightarrow d$ and $f_2 \in d_2 \Rightarrow d$, there is exactly one *strict* continuous function $[f_1, f_2]$ such that

$$\begin{tikzcd} d_1 \arrow[r, "\iota_1"] \arrow[dr, "f_1"'] & (d_1 + d_2)_\perp \arrow[d, "{[f_1, f_2]}"] & d_2 \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\ & d & \end{tikzcd}$$

It is easy to see that $(- + -)_\perp$ can be extended to a functor, and generalized to k arguments.

$\mathcal{T}$-Algebras

Let $\mathcal{T}$ be an endofunctor on the category of domains and *strict* continuous functions. Then a $\mathcal{T}$-algebra A consists of:

- A domain $|A|$ called the *carrier*,
- A single operation σ^A, which is a strict continuous function from $\mathcal{T}|A|$ to $|A|$.

When A and B are $\mathcal{T}$-algebras, a *homomorphism from A to B* is a strict continuous function ψ from $|A|$ to $|B|$ such that

$$\begin{tikzcd}
\mathcal{T}|A| \arrow[r, "\mathcal{T}\psi"] \arrow[d, "\sigma^A"] & \mathcal{T}|B| \arrow[d, "\sigma^B"] \\
|A| \arrow[r, "\psi"] & |B|
\end{tikzcd}$$

From m$\mathcal{T}$-Algebras to $\mathcal{T}$-Algebras

We can map m$\mathcal{T}$-algebras into $\mathcal{T}$-algebras as follows:

- The signature $\mathcal{T}_1, \ldots, \mathcal{T}_k$ is mapped into the endofunctor

$$\mathcal{T}(x) = (\mathcal{T}_1(x) + \cdots + \mathcal{T}_k(x))_\perp.$$

- Carriers $|A|$ are left unchanged.
- The operations $\sigma_1^A, \ldots, \sigma_k^A$ are mapped into

$$[\sigma_1^A, \ldots, \sigma_k^A].$$

- Homomorphisms are left unchanged.

It can be shown that this mapping carries initial algebras into initial algebras.

Initial $\mathcal{T}$-Algebras are Isomorphisms

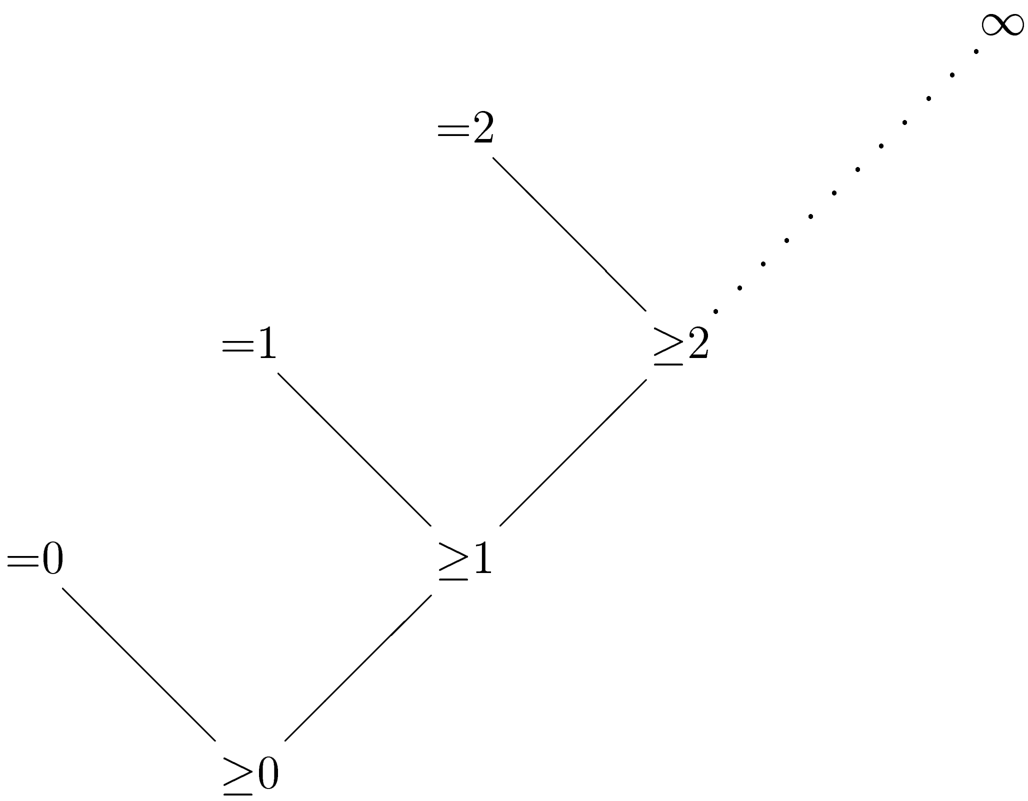
$$\begin{tikzcd}
\mathcal{T}|N| \arrow[r, "\mathcal{T}\phi_{\mathcal{T}|N|}"] \arrow[d, "\sigma^N"] & \mathcal{T}(\mathcal{T}|N|) \arrow[r, "\mathcal{T}\sigma^N"] \arrow[d, "\mathcal{T}\sigma^N"] & \mathcal{T}|N| \arrow[d, "\sigma^N"] \\
|N| \arrow[r, "\phi_{\mathcal{T}|N|}"] & \mathcal{T}|N| \arrow[r, "\sigma^N"] & |N|
\end{tikzcd}$$

- Since the right square commutes, σ^N is a homomorphism.
- By tne initiality of N, $\phi_{\mathcal{T}|N|}\,;\sigma^N = I_{|N|}$.
- Since the left square commutes,

  $$\sigma^N;\phi_{\mathcal{T}|N|} = \mathcal{T}\phi_{\mathcal{T}|N|};\mathcal{T}\sigma^N = \mathcal{T}(\phi_{\mathcal{T}|N|};\sigma^N) = \mathcal{T}I_{|N|} = I_{\mathcal{T}|N|}.$$

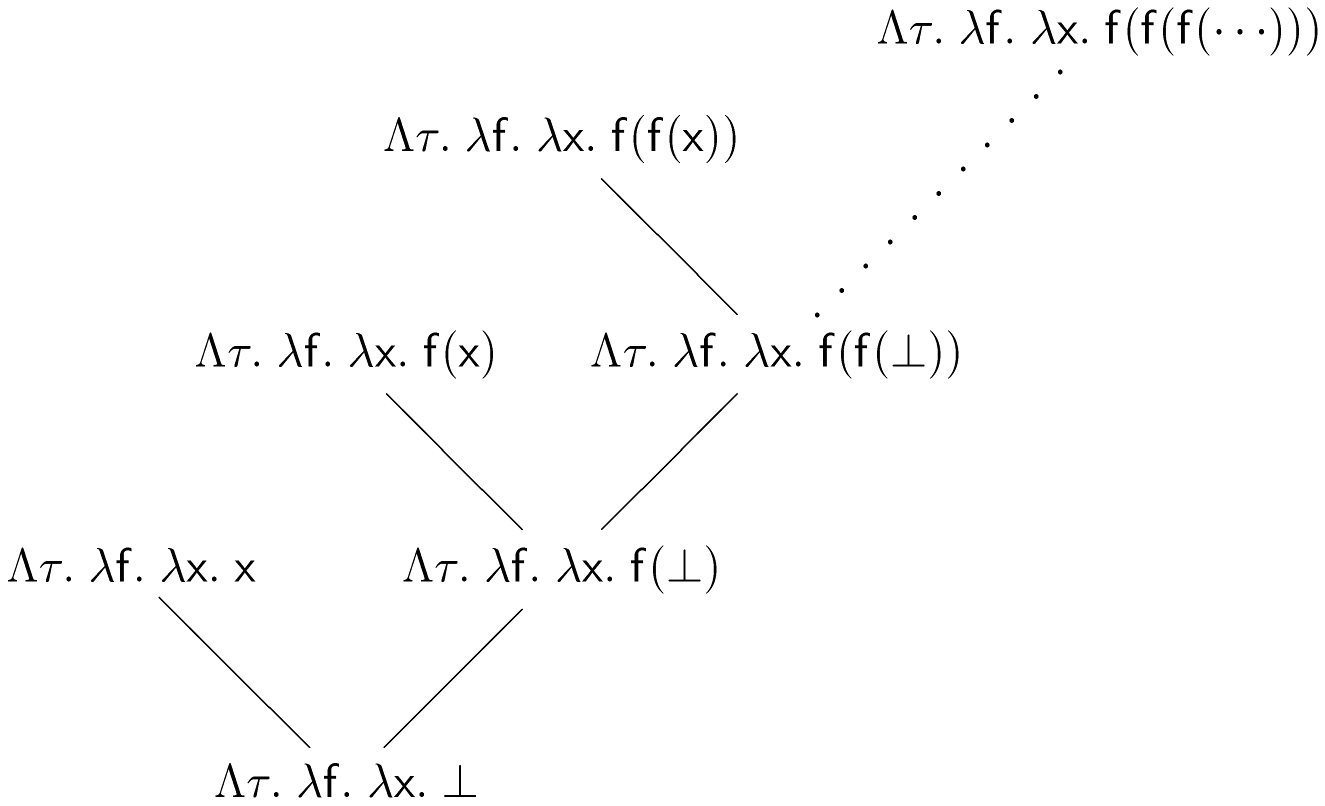
An Algebraic Polymorphic Domain

Thus we find that the meaning of $\forall\tau.\ (\tau \to \tau) \to \tau \to \tau$ is the initial m$\mathcal{T}$-algebra with one constant and one unary operator.



An Algebraic Polymorphic Domain

Thus we find that the meaning of $\forall\tau.\ (\tau \to \tau) \to \tau \to \tau$ is the initial m$\mathcal{T}$-algebra with one constant and one unary operator.



The General (One-Sorted) Case

The meaning of the polymorphic type

$$\forall\tau.\ (\underbrace{\tau \to \cdots \tau \to}_{n_1 \text{ times}} \tau) \to \cdots \to (\underbrace{\tau \to \cdots \tau \to}_{n_k \text{ times}} \tau) \to \tau$$

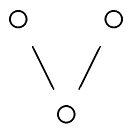
is the initial m$\mathcal{T}$-algebra with k operators of arities $n_1, \cdots, n_k$.

Instances

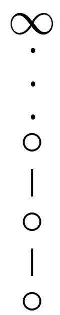
$$\forall\tau.\ \tau$$

$$\mathcal{O} = \forall\tau.\ \tau \to \tau$$

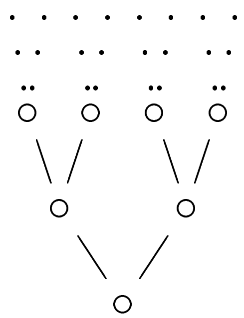
$$\forall\tau.\ \tau \to \tau \to \tau$$



$$\forall\tau.\ (\tau \to \tau) \to \tau$$



$$\forall\tau.\ (\tau \to \tau) \to (\tau \to \tau) \to \tau$$



Another Example

The domain of type

$$\forall\tau.\ (((\tau \to \mathcal{O}) \to \mathcal{O}) \to \tau) \to \tau$$

is the initial $\mathcal{T}$-algebra for

$$\mathsf{T}\tau = \Big((\tau \to \mathcal{O}) \to \mathcal{O}\Big)_{\perp},$$

where $\mathcal{O} = \forall\tau.\ \tau \to \tau$ denotes the domain

The initial algebra is: