Lecture 6: Introduction to transforms

This lecture is an introduction to the wonderful world of transforms. There is a lot that I'm not covering here, but this should be enough to get you started. Transforms are a hugely powerful technique that will allow you to easily get many moments of some random variable. For some problems, transforms are the only way we have of deriving these moments.

1 Solutions to homework problems from last class

1.1 Deriving second moment of delay for M/G/1 via RCL

Our goal is to derive $E[D^2]$ for the M/G/1 via RCL. The first insight is to realize that the $x'(t)$ function in RCL needs to be related to D^2 (what you want), which means that it may work to make $x(t)$ proportional to D^3.

The second insight is to realize that for any FIFO system, D, the delay witnessed by a customer, equals V, the work in the system at the time the customer arrives. But by PASTA (M/G/1), we have that $E[V^i]$ seen by a customer is also the time-average $E[V^i]$ in the system, averaged over all time. Hence we propose to make

$$x(t) = E[V^3].$$

But that implies that:

$$x'(t) = 3V(t)^2(-1)$$

Note that the above expression is valid assuming that $V(t) > 0$. However, it is also valid when $V(t) = 0$, since $x'(t)$ is then identically 0.

Given our expression for $x(t)$, it follows that jumps are defined as follows:

$$\begin{aligned}
-J_n &= (D_n + S_n)^3 - D_n^3 \\
&= 3D_n^2 S_n + 3D_n S_n^2 + S_n^3 \\
-E[J_n] &= 3E[D_n^2 S_n] + 3E[D_n S_n^2] + E[S_n^3] \\
-E[J_n] &= 3E[D_n^2]E[S_n] + 3E[D_n]E[S_n^2] + E[S_n^3] \\
-E[J] &= 3E[D^2]E[S] + 3E[D]E[S^2] + E[S^3]
\end{aligned}$$

Now RCL states that:

$$E[x'] = \lambda E[J]$$

where $E[x']$ is a notational convenience used to indicate the time-average: $\lim_{t\to\infty} \frac{1}{t}\int_0^t x'(s)ds$
Applying RCL, we have:

$$\begin{aligned}
E[x'] &= \lambda E[J] \\
3E[V^2] &= \lambda \left(3E[D^2]E[S] + 3E[D]E[S^2] + E[S^3]\right) \\
3E[D^2] &= \lambda \left(3E[D^2]E[S] + 3E[D]E[S^2] + E[S^3]\right) \\
E[D^2] &= \rho E[D^2] + \lambda E[D]E[S^2] + \frac{\lambda}{3}E[S^3] \\
(1-\rho)E[D^2] &= \lambda E[D]E[S^2] + \frac{\lambda}{3}E[S^3] \\
E[D^2] &= \frac{\lambda E[D]E[S^2]}{1-\rho} + \frac{\lambda}{3}\frac{E[S^3]}{1-\rho}
\end{aligned}$$

We now end, by substituting in our earlier result:

$$E[D] = \frac{\lambda E[S^2]}{2(1-\rho)}$$

which yields:

$$E[D^2] = \frac{1}{2}\left(\frac{\lambda E[S^2]}{1-\rho}\right)^2 + \frac{\lambda}{3}\frac{E[S^3]}{1-\rho}$$

1.2 Deriving Little's Law from RCL

Our goal is to derive $E[N] = \lambda E[T]$, when starting with $E[x'] = \lambda E[J]$.

The trick is to observe that we need $E[T]$ to take the place of $E[J]$. That is we need a jumpsize of

$$-J_n = T_n.$$

Once we've to this piece of intuition, it's clear that $x(t)$ needs to be defined so that it has a built-in jump of size T_n at arrival point a_n. At the same time, we want $x(t)$ to be continuous, outside of these countable number of jump points.

Let:

$$x(t) = \sum_{\text{n: in system at } t} \int_t^\infty I_n(s)ds$$

where

$$I_n(s) = \begin{cases} 1 & \text{if } C_n \text{ is in system at time } s \\ 0 & \text{otherwise} \end{cases}$$

Thus $x(t)$ represents the sum of the *remaining time in system* of all those jobs in the system at time t. Visually, $x(t)$ looks like Figure 1 below:

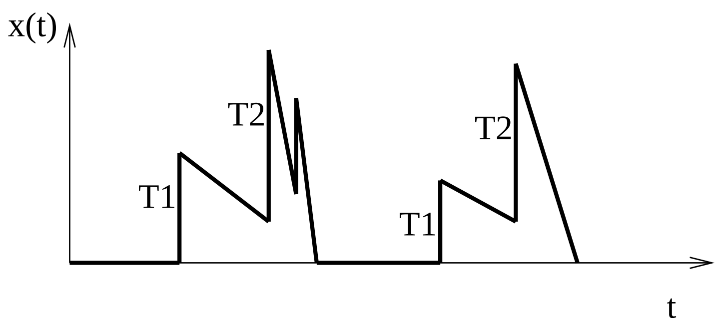

Figure 1: Illustration of $x(t)$ for deriving Little's Law from RCL.

Observe that $x(t)$ decreases continously at rate (slope) $-N(t)$, where $N(t)$ is the number of jobs in the system at time t, however it jumps up by T_n at time a_n. In a sense, we're done because we know that $x'(t)$ (the slope) is $-N(t)$. However we can be more formal as follows:

$$
\begin{aligned}
x(t) &= \sum_{\text{n: in system at } t} \int_t^{\infty} I_n(s)ds \\
x'(t) &= \frac{d}{dt} \sum_{\text{n: in system at } t} \int_t^{\infty} I_n(s)ds \\
x'(t) &= \sum_{\text{n: in system at } t} \frac{d}{dt} \int_t^{\infty} I_n(s)ds \\
&= \sum_{\text{n: in system at } t} \frac{d}{dt} \left(1 - \int_0^t I_n(s)ds\right) \\
&= \sum_{\text{n: in system at } t} -I_n(t) \\
&= -N(t) \\
E[x'] &= -E[N]
\end{aligned}
$$

We have assumed that we can bring the derivative inside the summation. This is possible because it's a finite sum, namely the number in system at time t is finite because in the statement of Little's Law, we're allowed to assume that the arrival rate equals the completion rate, i.e., the queue length is never infinite.

Together with

$$E[J] = -E[T],$$

we have proven Little's Law via RCL:

$$-E[N] = E[x'] = \lambda E[J] = -\lambda E[T]$$

1.3 Intuition regarding using RCL

All RCL papers seem to agree unanimously that there's no intuition to using RCL. I have tried to illustrate above that we can sometimes get clues by either:

1. Thinking about what $x(t)$ needs to look like, given the quantity we're trying to derive, and then using $x(t)$ to tell us what $E[J]$ looks like (this is the approach in the first example above), or
2. By thinking about what $E[J]$ needs to look like, and then using $E[J]$ to determine a function $x(t)$ that has those built-in jumps.

I agree that this method is very algebraic, and it's much harder to think about pictures, as we did in $\overline{H} = \lambda \overline{G}$.

2 Definitions of transforms and some examples

Definition 1 *The* **Laplace Transform** $L_f(s)$ *of a continuous function (on positive real axis)* $f(t)$ *is defined as*

$$L_f(s) = \int_0^\infty e^{-st} f(t) dt$$

where we will assume $s \geq 0$ *for convergence purposes.*

When we speak of the Laplace Transform of a continuous R.V. X, we are referring to the Laplace transform $L_f(s)$, of the p.d.f. f associated with X. Often we will write $\tilde{X}(s)$ to denote the Laplace transform of X.

Observe that if X is a continuous R.V. and f is the p.d.f. of X, then

$$\tilde{X}(s) = L_f(s) = E[e^{-sX}]$$

Example: Consider the Laplace Transform of $X = Exp(\lambda)$:

$$\begin{aligned} \tilde{X}(s) = L_f(s) &= \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt \\ &= \lambda \int_0^\infty e^{-(\lambda+s)t} dt \\ &= \frac{\lambda}{\lambda + s} \end{aligned}$$

Definition 2 *The* **z-Transform** $G_p(z)$ *of a discrete function (on non-negative integers)* $p(i)$ *is defined as*

$$G_p(z) = \sum_{i=0}^{\infty} p(i) z^i$$

where we assume $|z| \leq 1$ *for convergence purposes.*

When we speak of the z-transform of a discrete R.V. X, we are referring to the z-transform of the p.m.f. associated with X. Usually we will write $\hat{X}(z)$ to denote the z-transform of X.

Observe that if X is a discrete R.V. and $p(i)$ is its p.m.f., then

$$\hat{X}(z) = G_p(z) = E[z^X]$$

Example: Let A_t denote the number of arrivals in time t, where the arrival process is $Poisson(\lambda)$. Then, the z-transform of A_t is given by

$$\begin{aligned}
\hat{A}_t(z) = G_p(z) &= \sum_{i=0}^{\infty} \frac{(\lambda t)^i e^{-\lambda t} z^i}{i!} \\
&= e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t z)^i}{i!} \\
&= e^{-\lambda t} \cdot e^{(\lambda t) z} \\
&= e^{-\lambda t (1-z)}
\end{aligned}$$

Example: Let A_S denote the number of arrivals during one service time, where service time is denoted by S and has p.d.f. $f(t)$. Let a_i denote the probability that there rae i arrivals in time S. That is:

$$a_i = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} f(t) dt$$

Then

$$\begin{aligned}
\hat{A}_S(z) &= \sum_{i=0}^{\infty} a_i z^i \\
&= \sum_{i=0}^{\infty} \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} f(t) dt z^i \\
&= \int_0^{\infty} e^{-\lambda t} f(t) \sum_{i=0}^{\infty} \frac{(\lambda t z)^i}{i!} dt \\
&= \int_0^{\infty} e^{-\lambda t} f(t) e^{\lambda t z} dt \\
&= \int_0^{\infty} e^{-\lambda (1-z) t} f(t) dt \\
&= L_S(\lambda(1-z)) \\
&= \tilde{S}(\lambda(1-z))
\end{aligned}$$

(We'll see a quicker way to derive this soon.)

Example: Let X have p.m.f. of the form $p(i) = c \cdot \alpha^i$ for some constants c and α. Then the z-transform of X is given by

$$\begin{aligned}
\hat{X}(z) = G_p(z) &= \sum_{i=0}^{\infty} c \cdot \alpha^i z^i \\
&= c \sum_{i=0}^{\infty} (\alpha z)^i \\
&= \frac{c}{1 - \alpha z}
\end{aligned}$$

3 Getting moments from transforms

Theorem 1 *Let X be a continuous R.V. with p.d.f. $f(t)$. Then*

$$E[X^n] = (-1)^n \frac{d^n L_f(s)}{ds} \bigg|_{s=0}$$

Proof:

$$\begin{aligned}
e^{-st} &= 1 - (st) + \frac{(st)^2}{2!} - \frac{(st)^3}{3!} + \dots \\
e^{-st} f(t) &= f(t) - (st)f(t) + \frac{(st)^2}{2!} f(t) - \frac{(st)^3}{3!} f(t) + \dots \\
L_f(s) = \int_0^\infty e^{-st} f(t) dt &= \int_0^\infty f(t)dt - \int_0^\infty (st) f(t) dt + \int_0^\infty \frac{(st)^2}{2!} f(t) dt - \int_0^\infty \frac{(st)^3}{3!} f(t) dt + \dots \\
&= 1 - sE[X] + \frac{s^2}{2!} E[X^2] - \frac{s^3}{3!} E[X^3] + \dots \\
\frac{dL_f(s)}{ds} &= -E[X] + sE[X^2] - 3\frac{s^2}{3!} E[X^3] + \dots \\
\frac{dL_f(s)}{ds}\bigg|_{s=0} &= -E[X]
\end{aligned}$$

$$\frac{d^2 L_f(s)}{ds} = E[X^2] - sE[X^3] + \dots$$

$$\left.\frac{d^2 L_f(s)}{ds}\right|_{s=0} = E[X^2]$$

$$etc.$$

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Question: How do we know that the Laplace Transform as defined necessarily converges?

Answer (partial): It does provided f(t) is a p.d.f.. To see this observe that

$$e^{-t} \leq 1$$

for all non-negative values of t. Thus

$$e^{-st} = \left(e^{-t}\right)^s \leq 1$$

assuming that s is non-negative. Thus:

$$L_f(s) = \int_0^\infty e^{-st} f(t)dt \leq \int_0^\infty 1 \cdot f(t)dt = 1$$

assuming that s is non-negative.

Observation 1 *Observe that in the previous result, $f(t)$ did not have to be a p.d.f.. In particular, for any function $f(t)$,*

$$\int_{t=0}^\infty t^n \cdot f(t)dt = (-1)^n \left.\frac{d^n L_f(s)}{ds}\right|_{s=0}$$

Theorem 2 *Let X be a discrete R.V. with p.m.f. $p(i)$. Then the sequence:*

$$\{ \left.G_p^{(n)}(z)\right|_{z=1} : n \geq 1\}$$

provides the moments of X*, as follows:*

$$\begin{aligned} G'_p(z)|_{z=1} &= E[X] \\ G''_p(z)|_{z=1} &= E[X^2] - E[X] \\ G'''_p(z)|_{z=1} &= E[X^3] - 3E[X^2] + 2E[X] \end{aligned}$$

Proof:

$$\begin{aligned} G_p(z) &= p(0) + z^1 p(1) + z^2 p(2) + z^3 p(3) + z^4 p(4) + \cdots \\ G'_p(z) &= 1p(1) + 2zp(2) + 3z^2 p(3) + 4z^3 p(4) + \cdots \\ So, G'_p(1) &= E[X] \\ G''_p(z) &= 2p(2) + 3 \cdot 2 \cdot zp(3) + 4 \cdot 3 \cdot z^2 p(4) + \cdots \\ G''_p(z) &= (2^2 - 2)p(2) + (3^2 - 3)zp(3) + (4^2 - 4)z^2 p(4) + \cdots \\ So, G''_p(1) &= E[X^2] - E[X] \end{aligned}$$

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Example: Compute the first moment of A_S, the number of arrivals during a service time, where $S \sim Exp(\mu)$.

$$\begin{aligned} \hat{A_S}(z) &= \tilde{S}(\lambda(1-z)) \\ &= \frac{\mu}{\mu + \lambda(1-z)} \\ \hat{A_S}'(1) &= \frac{\lambda}{\mu} \checkmark \end{aligned}$$

Here's another way to do it, that is helpful to understand: This method is based on the chain rule.

$$E[A_S] = \hat{A_S}'(z) \mid_{z=1}$$

$$
\begin{aligned}
&= \frac{d}{dz}\hat{A}_S(z) \mid_{z=1} \\
&= \left(\frac{d}{dz}\tilde{S}(s) \text{ where } s = \lambda(1-z)\right) \mid_{z=1} \\
&= \left(\frac{d}{ds}\tilde{S}(s) \cdot \frac{ds}{dz}\right) \mid_{z=1} \\
&= -E[S] \cdot (-\lambda) \\
&= \frac{\lambda}{\mu}
\end{aligned}
$$

4 Linearity of Transforms

Theorem 3 *Let X and Y be continuous independent random variables with density functions x and y respectively. Let $Z = X + Y$. Then the Laplace transform of Z is given by $\tilde{Z}(s) = \tilde{X}(s) \cdot \tilde{Y}(s)$.*

In particular, if $X_1, \ldots, X_n$ are i.i.d random variables, and $Z = X_1 + \cdots + X_n$, then $\tilde{Z}(s) = (\tilde{X}(s))^n$.

Observe also that the Laplace transform of the convolution of x and y, $L_{x \otimes y}$, is equal to $L_x(s)L_y(s)$ even when X and Y are not *independent. Here $g(t) = x \otimes y(t) = \int_0^t x(t-k)y(k)dk$*

Proof:

$$
\begin{aligned}
\tilde{Z}(s) = L_z(s) &= \int_0^\infty e^{-st} z(t)dt \\
&= \int_0^\infty e^{-st} P(X+Y=t)dt \\
&= \int_0^\infty e^{-st} \int_{k=0}^t P(X+Y=t|Y=k) \cdot P(Y=k)dkdt \\
&= \int_0^\infty e^{-st} \int_{k=0}^t P(X=t-k|Y=k) \cdot y(k)dkdt \\
&= \int_0^\infty e^{-st} \int_{k=0}^t P(X=t-k) \cdot y(k)dkdt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty e^{-st} \int_{k=0}^t x(t-k) \cdot y(k) dk dt \\
&= \int_{k=0}^\infty y(k) \int_{t=k}^\infty e^{-st} x(t-k) dt dk \\
&= \int_{k=0}^\infty y(k) e^{-sk} \int_{t=k}^\infty e^{-s(t-k)} x(t-k) dt dk
\end{aligned}
$$

now, letting $v = t - k$, $dv = dt$, we have

$$
\begin{aligned}
&= \int_{k=0}^\infty y(k) e^{-sk} \int_{v=0}^\infty e^{-sv} x(v) dv dk \\
&= L_y(s) \cdot L_x(s)
\end{aligned}
$$

Observe that independence of X and Y was necessary to get from the 4th to the 5th line. Independence is unnecessary for the convolution function because there we're already at the 6th line. ■

Proof: [Alternative proof]

$$
\begin{aligned}
\tilde{Z}(s) &= E[e^{-sZ}] \\
&= E[e^{s(X+Y)}] \\
&= E[e^{sX} \cdot e^{sY}] \\
&= E[e^{sX}] \cdot E[e^{sY}] \\
&= \tilde{X}(s) \cdot \tilde{Y}(s)
\end{aligned}
$$

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Theorem 4 *Let X and Y be discrete independent random variables. Let $Z = X + Y$. Then the z-transform of Z is given by $\hat{Z}(z) = \hat{X}(z) \cdot \hat{Y}(z)$.*

Proof:

$$\begin{aligned}
G_Z(z) &= E[z^Z] \\
&= \sum_{n=0}^{\infty} P(Z=n) \cdot z^n \\
&= \sum_{n=0}^{\infty} P(X+Y=n) \cdot z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} P(X=k) \cdot P(Y=n-k|X=k) \cdot z^k \cdot z^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} P(X=k) \cdot P(Y=n-k) \cdot z^k \cdot z^{n-k} \\
&= \sum_{k=0}^{\infty} z^k P(X=k) \sum_{n=k}^{\infty} P(Y=n-k) z^{n-k} \\
&= \sum_{k=0}^{\infty} z^k P(X=k) \sum_{m=0}^{\infty} P(Y=m) z^m \\
&= \hat{X}(z) \cdot \hat{Y}(z)
\end{aligned}$$

■

Proof: [Alternative proof]

$$\begin{aligned}
\hat{Z}(z) &= E[z^Z] \\
&= E[z^{X+Y}] \\
&= E[z^X \cdot z^Y] \\
&= E[z^X] \cdot E[z^Y] \\
&= \hat{X}(z) \cdot \hat{Y}(z)
\end{aligned}$$

■

Theorem 5 *Let X, A, B be continuous random variables where*

$$X = \begin{cases} A & \textit{with probability } p \\ B & \textit{with probability 1 - } p \end{cases}$$

Let

$$\begin{aligned} x &= p.d.f.\ \ of\ X \\ a &= p.d.f.\ \ of\ A \\ b &= p.d.f.\ \ of\ B \end{aligned}$$

Then

$$\tilde{X}(s) = p \cdot \tilde{A}(s) + (1-p) \cdot \tilde{B}(s)$$

Proof:

$$\begin{aligned} \tilde{X}(s) = L_x(s) &= \int_0^\infty e^{-st} x(t) dt \\ \text{But } x(t) &= p \cdot a(t) + (1-p) \cdot b(t) \\ \text{So } L_x(s) &= p \int_0^\infty e^{-st} a(t) dt + (1-p) \int_0^\infty e^{-st} b(t) dt \\ &= p\tilde{A}(s) + (1-p)\tilde{B}(s) \end{aligned}$$

■

Example:

We can use a generalization of the above results to compute $\hat{A}_S(z)$ by conditioning:

$$\begin{aligned} \hat{A}_S(z) &= \int_0^\infty \hat{A}_S(z \mid S = t) f_S(t) dt \\ &= \int_0^\infty \hat{A}_t(z) f_S(t) dt \\ &= \int_0^\infty e^{-\lambda(1-z)t} f_S(t) dt \\ &= \tilde{S}(\lambda(1-z)) \end{aligned}$$

Theorem 6 *Let X, A, B be discrete random variables where*

$$X = \begin{cases} A & \textit{with probability } p \\ B & \textit{with probability } 1 - p \end{cases}$$

So

$$P(X = t) = P(A = t) \cdot p + P(B = t) \cdot (1 - p)$$

Then

$$\hat{X}(z) = p \cdot \hat{A}(z) + (1 - p) \cdot \hat{B}(z)$$

Proof:

$$\begin{aligned} \hat{X}(z) = G_X(z) &= E[z^X] \\ &= E[z^X | X = A] \cdot p + E[z^X | X = B] \cdot (1 - p) \\ &= E[z^A] \cdot p + E[z^B] \cdot (1 - p) \end{aligned}$$

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5 Combining Laplace and z-transforms

Theorem 7 [Summing Random number of independent random variables] *Let $Z = Y_1 + Y_2 + \ldots + Y_X$, where the Y_i's are i.i.d. continuous R.V.'s, and where X is a discrete random variable. Let $\hat{X}(z)$ be the* z-transform *of X, and let $\tilde{Y}(s)$ be the* Laplace transform *of Y. Then*

$$\tilde{Z}(s) = \hat{X}(\tilde{Y}(s))$$

Example We will apply the above theorem to prove that the sum of an $Geometric(p)$ number of independent exponentially distributed random variables with parameter μ is an exponentially distributed random variable with parameter $p\mu$:

First, observe that the *z-transform* of a geometrically distributed random variable is:

$$\hat{X}(z) = \sum_{n=1}^{\infty} p(1-p)^{n-1} z^n = \frac{zp}{1-(1-p)z}$$

Next, observe that the *Laplace-transform* of an exponentially distributed random variable is:

$$\tilde{Y}(s) = \frac{\mu}{s+\mu}$$

Thus,

$$\begin{aligned} \hat{X}(\tilde{Y}(s)) &= \frac{\frac{\mu}{s+\mu} p}{1-(1-p)\frac{\mu}{s+\mu}} \\ &= \frac{p\mu}{s+\mu-(1-p)\mu} \\ &= \frac{p\mu}{s+p\mu} \end{aligned}$$

which we recognize as the Laplace Transform for $Exp(p\mu)$.

Proof: Let $\tilde{Z}(s|n)$ denote the Laplace transform of $Z|_{\{X=n\}}$. Then $\tilde{Z}(s|n) = (\tilde{Y}(s))^n$. Now

$$\begin{aligned} \tilde{Z}(s) &= \sum_{n=0}^{\infty} Pr[X=n]\tilde{Z}(s|n) \\ &= \sum_{n=0}^{\infty} Pr[X=n](\tilde{Y}(s))^n \\ &= \hat{X}(\tilde{Y}(s)) \end{aligned}$$

■

6 More results on transforms

Theorem 8 *Let X and Y be random variables:*

$$L_{X(Y)}(s) = \int_{y=0}^{\infty} Pr[Y = y] \cdot L_{X(y)}(s)$$

Proof:

$$\begin{aligned} L_{X(Y)}(s) &= \int_{t=0}^{\infty} e^{-st} Pr[X(Y) = t] \\ &= \int_{t=0}^{\infty} e^{-st} \int_{y=0}^{\infty} Pr[X(y) = t] \cdot Pr[Y = y] \\ &= \int_{y=0}^{\infty} Pr[Y = y] \int_{t=0}^{\infty} e^{-st} Pr[X(y) = t] \\ &= \int_{y=0}^{\infty} Pr[Y = y] \cdot L_{X(y)}(s) \end{aligned}$$

■

Normally we look at Laplace transforms of p.d.f.s, but we could also ask what is the Laplace transform of any function, e.g., the c.d.f.:

Theorem 9 *Consider a p.d.f. $b(t)$, where $B(x)$ is the cumulative distribution function corresponding to $b(t)$. That is,*

$$B(x) = \int_0^x b(t)dt$$

Let

$$\tilde{b}(s) = L_{b(t)}(s) = \int_0^{\infty} e^{-st} b(t)dt$$

Let

$$\tilde{B}(s) = L_{B(x)}(s) = \int_0^{\infty} e^{-sx} \int_0^x b(t)dtdx$$

Then:

$$\tilde{B}(s) = \frac{\tilde{b}(s)}{s}$$

Proof:

$$
\begin{aligned}
\tilde{B}(s) &= \int_{x=0}^{\infty} e^{-sx} \int_{t=0}^{x} b(t)dtdx \\
&= \int_{x=0}^{\infty} e^{-st} \cdot e^{-s(x-t)} \int_{t=0}^{x} b(t)dtdx \\
&= \int_{t=0}^{\infty} b(t)e^{-st}dt \int_{x=t}^{\infty} e^{-s(x-t)}dx \\
&= \int_{t=0}^{\infty} b(t)e^{-st}dt \int_{y=0}^{\infty} e^{-sy}dy \\
&= \int_{t=0}^{\infty} b(t)e^{-st}dt \cdot \frac{1}{s} \\
&= \frac{\tilde{b}(s)}{s}
\end{aligned}
$$

■

One last little bit of information which comes in very handy when differentiating Laplace transforms and z-transforms:

Theorem 10

$$\tilde{X}(0) = 1, \ \forall \ \textit{random variables } X$$

$$\hat{X}(1) = 1, \ \forall \ \textit{random variables } X$$

7 An example: Deriving the Laplace Transform of the Busy Period of an M/G/1 queue

We are interested in studying the length of a busy period in an M/G/1 queue. Let B denote the length of a busy period.

Question: Recall that last semester we derived $E[B]$. How did we do that?

Answer: We invoked Renewal Reward theory.

Suppose we earn money at a rate of \$1 for each unit of time that the server is busy. Let $R(t)$ denote the reward earned by time t. Then the average rate at which we earn money is:

$$\lim_{t\to\infty} \frac{R(t)}{t} = \text{Fraction of time server is busy} = \rho$$

Now we look at the same problem from a Renewal Reward perspective. We say that a "renewal" occurs whenever the system transitions from being idle to being busy. The length of a renewal period is then $B + I$, where B denotes the length of a busy period and I denotes the length of an idle period. By Renewal-Reward theory, the average rate at which we earn money is

$$\begin{aligned}
\lim_{t\to\infty} \frac{R(t)}{t} &= \frac{\text{Expected reward earned during renewal period}}{\text{Expected length of renewal period}} \\
&= \frac{E[B]}{E[B+I]} \\
&= \frac{E[B]}{E[B]+E[I]}
\end{aligned}$$

Equating the two ways of viewing the average rate at which we earn money, and observing that $E[I] = 1/\lambda$, we have:

$$\begin{aligned}
\rho &= \frac{E[B]}{E[B]+E[I]} \\
E[B] &= \rho(E[B]+E[I]) \\
E[B] &= \rho E[B] + E[S] \\
E[B] &= \frac{E[S]}{1-\rho}
\end{aligned}$$

We now ask: How do we get the second moment of the busy period? Or the third moment of the busy period?

To do this, we will need to use Laplace transforms.

B denotes the length of a busy period, and $B(x)$ is the length of a busy period that starts with a job of size x. Let $\widetilde{B}(s)$ and $\widetilde{B(x)}(s)$ denote the corresponding Laplace transforms.

Let S denote the r.v. for the service time, and $\widetilde{S}(s)$ denote the corresponding Laplace transform.

Our approach is a little complicated. What we'll do if first form an expression for $B(x)$ in terms of B. Then we'll compute the Laplace transform of $B(x)$ as a function of the Laplace transform of B. We will then get the Laplace transform of B by unconditioning the Laplace transform of $B(x)$. This is going to seem weird. In the end, we will have the Laplace transform of B defined in terms of itself. This will seem even weirder. However, it's not a problem, because when we differentiate this final expression, all the moments will fall out.

If we define A_x to be the number of arrivals in a time segment of length x, then we have that:

$$B(x) = x + \sum_{i=1}^{A_x} B_i$$

where each B_i is distributed identically as B.

Using the above equation,

$$\begin{aligned} \widetilde{B(x)}(s) &= \widetilde{x} \;\; . \;\; \widetilde{(\sum_{i=1}^{A_x} B_i)} \\ &= e^{-sx} \;\; . \;\; \widehat{A_x}(\widetilde{B}(s)) \end{aligned}$$

Now we know that

$$\widehat{A_x}(z) = e^{-\lambda x(1-z)}$$

So,

$$\widehat{A_x}(\widetilde{B}(s)) = e^{-\lambda x(1-\widetilde{B}(s))}$$

And so,

$$\widetilde{B(x)}(s) = e^{-sx}.e^{-\lambda x(1-\widetilde{B}(s))} = e^{-x(s+\lambda-\lambda\widetilde{B}(s))}$$

To get $\widetilde{B}(s)$ from $\widetilde{B(x)}(s)$, we just uncondition as below, where $f(\cdot)$ denotes the probability density function of S.

$$\begin{aligned}\widetilde{B}(s) &= \int_0^\infty \widetilde{B(x)}(s) f(x) dx \\ &= \int_0^\infty e^{-x(s+\lambda-\lambda\widetilde{B}(s))} f(x) dx \\ &= \widetilde{S}(s+\lambda-\lambda\widetilde{B}(s))\end{aligned}$$

Thus we've shown that:

$$\widetilde{B}(s) = \widetilde{S}(s+\lambda-\lambda\widetilde{B}(s))$$

The first moment $E[B]$ is given by

$$\begin{aligned}E[B] = -\widetilde{B}'(s)|_{s=0} &= -\widetilde{S}'(s+\lambda-\lambda\widetilde{B}(s))|_{s=0}\ \ (1-\lambda\widetilde{B}'(s)|_{s=0}) \\ &= -\widetilde{S}'(0+\lambda-\lambda(1))(1+\lambda E[B]) \\ &= -\widetilde{S}'(0)(1+\lambda E[B]) \\ &= E[S](1+\lambda E[B])\end{aligned}$$

Solving for $E[B]$, we get

$$E[B] = \frac{E[S]}{1-\lambda E[S]}$$

To get the second moment, we differentiate $\widetilde{B}'(s)$ again and evaluate the result at $s = 0$. This yields

$$\begin{aligned}E[B^2] = \widetilde{B}''(s)|_{s=0} &= \frac{d}{ds}[\widetilde{S}'(s+\lambda-\lambda\widetilde{B}(s))(1-\lambda\widetilde{B}'(s)]|_{s=0} \\ &= \widetilde{S}''(0)[1-\lambda\widetilde{B}'(0)]^2 + [(\widetilde{S}'(0))(-\lambda\widetilde{B}''(0))] \\ &= E[S^2][1+\lambda E[B]]^2 + \lambda E[S]E[B^2]\end{aligned}$$

Substituting for $E[B]$ and solving for $E[B^2]$ we get

$$E[B^2] = \frac{E[S^2]}{(1-\lambda E[S])^3}$$

8 Readings on Transforms

Many books have at least a section on transforms. The following is a whole book on transforms. The book is in the library.

Transform Techniques in Probability Modeling, Walter C. Giffin, Academic Press, 1975.

9 Homework Problems for Next Time

Here are the homework problems that I'd like you to work on. As before, the derivations should be quite short.

1. **[Response Time for M/M/1]** Last semester we derived the mean response time for an M/M/1 queue. We also derived the distribution on the number of jobs in the M/M/1.

   Let T denote the response time of an M/M/1. In this problem you will derive $\widetilde{T}(s)$. This will allow you to answer the question: What is the distribution of T?

   Some hints: (i) You will need to condition on the number of jobs seen by an arrival. (ii) You will need to invoke the PASTA principle. (iii) When you get the final answer, you will recognize the transform.

2. **[Number of Jobs Served During M/M/1 Busy Period]** Let N_B denote the number of jobs served during an M/M/1 busy period.

   (a) Derive $E[N_B]$.

   (b) Derive the z-transform: $\widehat{N_B}(z)$. Determine the first and second moments of N_B by carefully differentiating your transform. (Note, if you instead determine the Laplace-transform: $\widetilde{N_B}(s)$, the moments you get should look the same.)

Thank you!