Spectral Graph Theory and its Applications November 5, 2004

Lecture 9

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9.1 Expanders

The topic of this lecture is expander graphs. I will explain how one can use bounds on μ_2 to prove expansion-like properties, and will also prove bounds on how good μ_2 can be.

9.2 Quasi-Random Properties

Let $G = (V, E)$ be a d-regular graph. For now, I want to consider how a random subset of the vertices of G looks. For example, we could choose a set $X \subseteq V$ by putting each vertex in A with probability α, independently for each vertex. If we do this, how many edges to we expect to find between vertices in X?

The answer is simple: for each edge in the graph, the probability it winds up in X is α^2. As there are $dn/2$ edges in the graph, we expect to find $\alpha^2 dn/2$ edges in A.

We will prove that in an expander graph, every set X of size αn contains approximately $\alpha^2 dn/2$ edges! So, every set looks like a random set. In fact, we will prove something even stronger.

What if we choose two sets X and Y at random, putting vertices in X with probability α and putting vertices in Y with probability β. I will make all these choices independently, so that X and Y can overlap. I can again ask how many edges I expect to find of the form (u, v) with $u \in X$ and $v \in Y$ and $u < v$. If $u \in X \cap Y$ and $v \in X \cap Y$, I will count the edge twice. Reasoning as before, we find that the answer is $\alpha\beta dn$. We will show that, in a good expander, this is the approximately the answer for all sufficiently large sets A and B.

To state the theorem, I use the notation

$$e(X, Y) = \{(u, v) \in X \times Y : (u, v) \in E\}.$$

Theorem 9.2.1. *Let $G = (V, E)$ be a d-regular graph on n nodes such that every eigenvalue but the largest has absolute value at most μ. Let $X, Y \subseteq V$ have sizes $|X| = \alpha n$ and $|Y| = \beta n$. Then,*

$$|e(X, Y) - \alpha\beta dn| \leq \mu n \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

This result is applicable when $\mu/d \leq \sqrt{\alpha\beta}$.

Proof of Theorem 9.2.1. This proof will follow from the standard tricks. We will let x be the characteristic vector of X and y be the characteristic vector of Y. We then observe that

$$x^T A y = e(X, Y).$$

To bound $x^T Ay$, we set $v = x - \alpha\mathbf{1}$ and $w = y - \beta\mathbf{1}$, so that v and w are orthogonal to $\mathbf{1}$. We can then compute

$$\begin{aligned} x^T Ay &= (v + \alpha\mathbf{1})^T A(w + \beta\mathbf{1}) \\ &= v^T Aw + \alpha\mathbf{1}^T Aw + \beta v^T A\mathbf{1} + \alpha\beta\mathbf{1}^T A\mathbf{1}. \end{aligned}$$

We now examine each of these terms. The easiest two are the middle terms: since $A\mathbf{1} = d\mathbf{1}$, and v is orthogonal to $\mathbf{1}$,

$$\beta v^T A\mathbf{1} = 0.$$

Similarly, we find that

$$\alpha\mathbf{1}^T Aw = 0.$$

For the last term, we compute

$$\alpha\beta\mathbf{1}^T A\mathbf{1} = \alpha\beta\mathbf{1}^T (d\mathbf{1}) = \alpha\beta dn.$$

So,

$$e(X, Y) - \alpha\beta dn = v^T Aw.$$

To bound the right-hand term in this equality, we note that $\|Aw\| \leq \mu \|w\|$ (using the same trick as we used last class), and so

$$\left|v^T Aw\right| \leq \|v\| \|Aw\| \leq \mu \|v\| \|w\| .$$

Finally, a routine calculation reveals that

$$\|v\| = \sqrt{n(\alpha - \alpha^2)} \quad \text{and} \quad \|w\| = \sqrt{n(\beta - \beta^2)},$$

so

$$\left|v^T Aw\right| \leq n\sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

□

9.3 Expansion

We will now derive a bound the most fundamental property of expander graphs: vertex expansion.

Theorem 9.3.1 (Tanner). *Let G be a d-regular graph with an adjacency matrix A in which every eigenvalue other than d has absolute value at most μ. Then, for every set $X \subseteq V$,*

$$|N(X)| \geq \frac{d^2 |X|}{\mu^2 + (d^2 - \mu^2) |X| / n}. \tag{9.1}$$

Proof. This theorem will follow quickly from Theorem 9.2.1. Let $Y = V - N(X)$, and set $|Y| = \beta n$. By construction $e(Y, X) = 0$, and $N(X) = (1 - \beta)n$. Applying Theorem 9.2.1, we find

$$\alpha\beta dn \leq \mu n\sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

After some simple manipulation, this inequality becomes

$$(1 - \beta) \geq \frac{d^2}{\mu^2 + (d^2 - \mu^2)\,|X|\,/n}.$$

□

I remark that this is not how Tanner originally proved this theorem: he instead considered the norm of Ax, and applied the Cauchy-Schwartz inequality to show that it must be non-zero in many places.

Let's examine how the right-hand side of (9.1) behaves for some interesting setting of the parameters. In a Ramanujan graph, $\mu \leq 2\sqrt{d-1}$. If we assume that $|X|\,/n$ is small, then we essentially get $|N(X)| \geq (d/4)\,|X|$. This is a very strong inequality, as we always have $|N(X)| \leq d\,|X|$.

Unfortunately, many applications of expander graphs require an expansion factor at least $(d/2)$ for small sets. There were both positive and negative developments in our attempts to achieve such expansion. Kahale improved Tanner's bound to show that for sufficiently small (but constant) α, one would obtain $|N(X)| \geq (d/2 - o(1))\,|X|$. On the other hand, Kalahe also showed that one could modify explicit constructions of expander graphs to obtain graphs with $\mu \leq 2\sqrt{d-1}$ yet with a set of two vertices with the same set of d neighbors, and so expansion factor at most $d/2$. This later result ended most attempts to achieve expansion factor $d/2$ through eigenvalue analysis.

Little progress was made until 2002, when Capalbo, Reingold, Wigderson and Vadhan came up with a new technique for constructing and analyzing expander graphs, and used this technique to prove that their graphs had expansion up to $d(1 - \epsilon)$ for sufficiently small sets. Their technique does not depend upon eigenvalues, so I will not explain it in this course.

9.4 Explicit Constructions

There isn't much that I can tell you about the explicit constructions of expanders, but I can tell you how they look. Margulis and, independently, Lubotzky, Phillips and Sarnak constructed d-regular Ramanujan graphs ($\mu \leq 2\sqrt{d-1}$) from Cayley graphs of the projective special linear groups over finite fields. In particular, let π be a prime congruent to 1 modulo 4, and let Z_π denote the integers modulo π. Our vertices will correspond to elements of $PSL(Z_\pi)$: the 2-by-2 matrices with determinant 1 in which we identify A and $-A$. A cayley graph on this vertex set is given by a set S of elements of $PSL(Z_\pi)$, by putting an edge between matrices A and B if $AB^{-1} \in S$.

In these constructions, S is determined by another prime p congruent to 1 modulo 4 that is a quadratic residue modulo π. We consider the solutions to the equation $a_1^2 + a_2^2 + a_3^2 + a_4^2 = p$, where a_1 is odd and a_2, a_3 and a_4 are even. One can show that there are $p + 1$ such solutions. For each, we put the following matrix in S:

$$\frac{1}{\sqrt{p}}\left[\begin{array}{cc} a_0 + i a_1 & a_2 + i a_3 \\ -a_2 + i a_3 & a_0 - i a_1 \end{array}\right],$$

where i satisfies $i^2 = -1$ modulo π.

What we learn from this discussion is that these matrices are rather concrete, and that we can easily perform computations such as determining the neighbors of a vertex. In particular, we can perform these computations in time polynomial in the length of the label of a vertex, and do not need to store the entire graph.

9.5 Lower bounds on μ_2

I will conclude the class by presenting a lower bound which shows that for every $\epsilon > 0$, for sufficiently large graphs, μ_2 cannot be lower than $2\sqrt{d-1} - \epsilon$. For the following proof, which is attributed to A. Nilli but which we suspect was written by N. Alon, we find it more convenient to work with the Laplacian.

Theorem 9.5.1. *Let G be a d-regular graph containing two edges (u_0, u_1) and (v_0, v_1) that are at distance at least $2k+2$. Then,*

$$\lambda_2 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k+1}.$$

Proof. Our proof will follow from the construction of a carefully chosen test vector. We first define sets

$$\begin{aligned} U_0 &= \{u_0, u_1\} \\ U_i &= N(U_{i-1}) - \cup_{j \leq i-1} U_j, \text{ for } i \leq k \\ V_0 &= \{v_0, v_1\} \\ V_i &= N(V_{i-1}) - \cup_{j \leq i-1} V_j, \text{ for } i \leq k. \end{aligned}$$

That is, U_i consists of the vertices at distance exactly i from U_0. Let $\bar{U} = \cup U_i$ and $\bar{V} = \cup V_i$

Note that there are no edges betweena $\bar{U}$ and $\bar{V}$.

For some constants α and β to be chosen momentarilly, we set

$$x(a) = \begin{cases} \frac{\alpha}{(d-1)^{-i/2}} & \text{for } a \in U_i \\ -\frac{\beta}{(d-1)^{-i/2}} & \text{for } a \in V_i \\ 0 & \text{otherwise.} \end{cases}$$

We now choose α and β so that x is orthogonal to the all-1s vector. It turns out that the choice is otherwise unimportant.

To aid in our evaluation of the Rayleigh quotient of x, let E_U denote the set of edges attached to vertices in $\bar{U}$, and define E_V analogously. From our assumption that U_0 and V_0 are at distance at least $2k+2$, we know that E_U and V_U are disjoint. Thus, the Rayleigh quotient is

$$\frac{\sum_{(a,b)\in E_U}(x(a)-x(b))^2+\sum_{(a,b)\in E_V}(x(a)-x(b))^2}{\sum_{a\in\bar{U}}x(a)^2+\sum_{a\in\bar{V}}x(a)^2+} \leq \max\left(\frac{\sum_{(a,b)\in E_U}(x(a)-x(b))^2+}{\sum_{a\in\bar{U}}x(a)^2+}, \frac{\sum_{(a,b)\in E_V}(x(a)-x(b))^2}{\sum_{a\in\bar{V}}x(a)^2}\right),$$

by my favorite inequality. We will just consider one of these terms, as they are symmetric. We first compute

$$\sum_{a\in\bar{V}}x(a)^2=\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}.$$

As each vertex $a\in U_i$ has at most $d-1$ neighbors in U_{i+1} (this is why we started from an edge rather than a vertex), we have

$$\begin{aligned}
\sum_{(a,b)\in E_V}(x(a)-x(b))^2 &\leq \sum_{i=0}^{k-1}|U_i|\,(d-1)\left(\frac{1}{(d-1)^{i/2}}-\frac{1}{(d-1)^{(i+1)/2}}\right)^2+|U_k|\,(d-1)\frac{1}{(d-1)^k}\\
&=\sum_{i=0}^{k-1}\frac{|U_i|}{(d-1)^i}\left(\sqrt{d-1}-1\right)^2+|U_k|\,\frac{1}{(d-1)^{k-1}}\\
&=\sum_{i=0}^{k-1}\frac{|U_i|}{(d-1)^i}\left(d-2\sqrt{d-1}\right)+|U_k|\,\frac{d-2\sqrt{d-1}}{(d-1)^k}\,|U_k|\,\frac{2\sqrt{d-1}-1}{(d-1)^k}\\
&=\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}\left(d-2\sqrt{d-1}\right)+|U_k|\,\frac{2\sqrt{d-1}-1}{(d-1)^i}.
\end{aligned}$$

Now, we clearly have

$$\frac{\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}\left(d-2\sqrt{d-1}\right)}{\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}.}\leq\left(d-2\sqrt{d-1}\right).$$

Finally, we observe that

$$\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}\geq(k+1)\frac{|U_k|}{(d-1)^k},$$

so

$$\frac{|U_k|\,\frac{2\sqrt{d-1}-1}{(d-1)^k}}{\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}}\leq\frac{2\sqrt{d-1}-1}{k+1}.$$

$\square$