6.1 Adjacency Matrices, Random Walks, and Expander Graphs

I have three goals for this lecture. The first is to introduce one of the most important familes of graphs: expander graphs. They are the source of much combinatorial power, and the counter-example to numerous conjectures. We will become acquainted with these graphs by examining random walks on them. To facilitate the analysis of random walks, we will examine these graphs through their adjacency matrices.

6.2 Expander Graphs

There are many ways of defining expander graphs. Most consider a sequence of graphs $(G_n)_n$, one for each n, rather than an individual graph. We say that a sequence of d-regular graphs $(G_n)_n$ is a *family of expander graphs* if there exists a constant c such that $\lambda_2(G_n) \geq c$ for all n.

I would draw you an expander, but the picture would be ugly. To see why, recall from Lecture 4 that

$$\lambda_2/2 \leq \phi(G) \stackrel{\text{def}}{=} \min_S \frac{e(S, \bar{S})}{\min\left(|S|, |\bar{S}|\right)}.$$

In this case, it tells us that every set of at most half of the vertices S has at least $(c/2)|S|$ edges leaving it. In particular, this means that the graph does not have any small cuts. On the other hand, if I could draw a nice picture of the graph, then it would have small cuts (I might make this a homework problem).

You should now be wondering: do families of expanders exist, and how large can c be if they do? Expanders do exist. It has been known for some time that a randomly chosen d-regular graphs are expanders with high probability. There are also explicit constructions. Assymptotically, c can be no larger than $d - 2\sqrt{d-1}$. Moreover, there are explicit constructions of infinite families of graphs, known as Ramanujan graphs, that achieve $\lambda_2(G) > d - 2\sqrt{d-1}$ for all graphs in the family.

It is often more convenient ot examine the spectra of the adjacency matrix of expanders, A. We will let $d = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ denote the eigenvalues of A, and remark that $\mu_i = d - \lambda_i$. Ramanujan graphs satisfy $|\mu_i| \leq 2\sqrt{d-1}$, for all $i > 1$.

6.3 Random Walks

I will now loosely explain the concept of a random walk on a graph. We will begin at some vertex, say 1. At the next time step, we will move to a random neighbor of that vertex. And, at the next time step, we will move to a random neighbor of that vertex. And so on.

Random walks on d-regular graphs are best understood by normalizing the adjacency matrix A to obtain the walk matrix $M = A/d$. We can then describe the probability distribution of our walk at time t as a vector. For example, if we start at vertex 1 at time 0, then the initial distribution is

$$p^0(i) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The distribution on vertices at time 1 is then given by $p^1 = Mp^0$, and at time t by

$$p^t = M^t p^0.$$

In this lecture and the next, we will see many properties of random walks. As we will often examine them through spectral techniques, I will let the eigenvalues of M be $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, and note that $\rho_i = \mu_i/d$.

We'll begin by considering random walks on expanders. We will prove that a random walk on an expander is unlikely to visit any small neighborhood a large number of times. To make the proof simple, I'll use some extreme parameters. This proof is from [IZ89]

Theorem 6.3.1. *Let $G = (V, E)$ be a d-regular expander graph with $|\mu_i|/d \leq 1/30$, for all $i \geq 1$. Let S be a set of vertices of size at most $|V|/36$. Consider the random walk that begins at a uniformly chosen vertex, and walks for k steps. Then, the probability the walk is in S more than half the time is at most*

$$\left(\frac{2}{\sqrt{5}}\right)^k.$$

Since our initial distribution is uniform, we will have

$$p^1 = (1/n, \ldots, 1/n).$$

We now define two diagonal matrices, X and Y, that will facilitate our analysis:

$$x_{i,j} = \begin{cases} 1 & \text{if } i = j \in S \\ 0 & \text{otherwise.} \end{cases}$$

$$y_{i,j} = \begin{cases} 1 & \text{if } i = j \notin S \\ 0 & \text{otherwise.} \end{cases}$$

To see how we use these, we observe the probability that a uniformly chosen vertex is in S is given by

$$\left\| X p^1 \right\|_1,$$

where we recall that for a vector v,

$$\|v\|_1 = \sum_i |v_i| .$$

Similarly, the probability that the random walk starts in S and in the second step is not in S is given by

$$\left\|YMXp^1\right\|_1 .$$

In general, for $T_i \in \left\{S, \bar{S}\right\}$, the probability that the walk is in T_i at the ith step for all i is

$$\left\|Z_k M Z_{k-1} M Z_{k-2} \dots Z_1 p^1\right\|_1 , \tag{6.1}$$

where

$$Z_i \stackrel{\text{def}}{=} \begin{cases} X & \text{if } T_i = S \\ Y & \text{if } T_i = \bar{S}. \end{cases}$$

Since $Mp^1 = p^1$, I will re-write (6.1) as

$$\left\|Z_k M Z_{k-1} M Z_{k-2} \dots Z_1 M p^1\right\|_1 .$$

So that we can exploit our knowledge of the eigenvalues, we will analyze this probability through 2-norms. We begin by recalling that for every length n vector

$$\|p\|_1 \leq n \|p\|_2 .$$

To see that this inequality is the best possible, note that $\left\|p^1\right\|_1 = 1$ while $\left\|p^1\right\|_2 = 1/n$.

The key to our analysis is the following lemma.

Lemma 6.3.2. *For all vectors p,*

$$\|XMp\| \leq \frac{\|p\|}{5}.$$

Proof. Write $p = \alpha\mathbf{1} + x$, where $x \perp \mathbf{1}$. As x and $\mathbf{1}$ are orthogonal, we have

$$\|p\| \leq \|\alpha\mathbf{1}\| + \|x\| .$$

We now have

$$XMp = \alpha XM\mathbf{1} + XMx.$$

We first note that $XM\mathbf{1} = X\mathbf{1}$, and

$$\|X\mathbf{1}\| = \sqrt{\frac{|S|}{n}} \|\mathbf{1}\| \leq \|\mathbf{1}\| /6,$$

and so

$$\|\alpha XM\mathbf{1}\| \leq \|\alpha\mathbf{1}\| /6. \tag{6.2}$$

Let $\mathbf{1} = v_1, v_2, \ldots, v_n$ be the eigenvectors of M corresponding to $\rho_1, \ldots, \rho_n$. We will now show that $\|Mx\| \leq (1/30)\|x\|$. As M is symmetric and so the eigenvectors are orthogonal, we have

$$\begin{aligned} Mx &= M\left(\sum_{i=1}^{n}(v_i^T x)v_i\right) \\ &= M\left(\sum_{i=2}^{n}(v_i^T x)v_i\right), \text{ as } v_1^T x = 0, \\ &= \rho_i \sum_{i=2}^{n}(v_i^T x)v_i. \end{aligned}$$

We then observe that

$$\begin{aligned} \|Mx\|_2^2 &= \left\|\rho_i \sum_{i=2}^{n}(v_i^T x)v_i\right\|_2^2 \\ &= \sum_{i=2}^{n}\rho_i \left\|(v_i^T x)v_i\right\|_2^2, \text{ as the } v_i\text{s are orthogonal} \\ &\leq (1/30)\sum_{i=2}^{n}\left\|(v_i^T x)v_i\right\|_2^2, \\ &= (1/30)\|x\|^2. \end{aligned}$$

Thus,

$$\|XMx\| \leq \|x\|/30.$$

Combining this inequality with (6.2), we obtain

$$\|XMp\| \leq \|\alpha\mathbf{1}\|/6 + \|x\|/30 \leq \|p\|/6 + \|p\|/30 \leq \|p\|/5.$$

□

We now return to the proof of the Theorem. From the lemma, we know that for any sequence $(T_i)_i$ in which at least half of the entries are S, we have

$$\left\|Z_k M Z_{k-1} M Z_{k-2} \ldots Z_1 M p^1\right\|_2 \leq \left\|p^1\right\|/5^{(k/2)} = 1/(\sqrt{n}5^{k/2}).$$

On the other hand,

$$\left\|Z_k M Z_{k-1} M Z_{k-2} \ldots Z_1 M p^1\right\|_1 \leq \sqrt{n}\left\|Z_k M Z_{k-1} M Z_{k-2} \ldots Z_1 M p^1\right\|_2,$$

and so

$$\left\|Z_k M Z_{k-1} M Z_{k-2} \ldots Z_1 M p^1\right\|_1 \leq 5^{-k/2}.$$

Summing this inequality over all such sequences of T_i, we obtain

$$\begin{aligned} \mathrm{P}\left[\text{the walks hits } S \text{ more than } k/2 \text{ times}\right] &= \sum_{\substack{T_1,\ldots,T_k \\ \#\{i:T_i=S\}\geq k/2}} \left\|Z_k M Z_{k-1} M Z_{k-2} \ldots Z_1 M p^1\right\|_1 \\ &\leq 2^k 5^{-k/2} \\ &= \left(\frac{2}{\sqrt{5}}\right)^{k/2}. \end{aligned}$$

References

[IZ89] R. Impagliazzo and D. Zuckerman. How to recycle random bits. In IEEE, editor, *30th annual Symposium on Foundations of Computer Science, October 30–November 1, 1989, Research Triangle Park, North Carolina*, pages 248–253, 1109 Spring Street, Suite 300, Silver Spring, MD 20910, USA, 1989. IEEE Computer Society Press.