Tutte Embeddings of Planar Graphs

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21.1 Tutte's Theorem

We usually think of graphs as being specified by vertices and edges. But, planar graphs have another fundamental unit: faces. Formally, a face in a planar graph is a minimal simple cycle of edges.

Tutte proved that if one takes a 3-connected planar graph, fixes the vertices of one face so that the vertices are the corners of a convex polygon, and the edges the edges, and then let every other vertex be the average of its neighbors, then one obtains a planar straight-line embedding of the graph.

The only property that we will use of 3-connected graphs in the proof is that for every 5 vertices u, v_0, v_1, s, t there are either

- vertex-disjoint paths from v_0 to s and v_1 to t that do not go through u, or
- vertex-disjoint paths from v_0 to t and v_1 to s that do not go through u.

I'll present a proof of Tutte's theorem from the paper "One-Forms on Meshes and Applications to 3D Mesh Parameterization" by Gortler, Gotsman and Thurston (Harvard CS TR-12-04).

The only fact about Tutte embeddings that we will use in the proof is that every non-boundary vertex is a strict convex combination of its neighbors. That is, is can be written as a sum of its neighbors with non-zero coefficients.

21.2 Planarity

Given some planar embedding of a planar graph, we can define an orientation of the edges of every face. By default, we will assume that the edges of all internal faces are oriented clockwise, and the external face is oriented counter-clockwise. Whenever we consider an edge at a face, we will consider it with its orientation.

A Tutte embedding assigns to each vertex u coordinates $(x(u), y(u))$. We let (G, x, y) denote a Tutte embedding. For a face f in the graph, we say that the embedding of f is *convex* if the polygon corresponding to the face is convex. That is, if $v_1, \ldots, v_k$ are the vertices on the face in order, then no horizontal line crosses more than two of the edges (v_{i-1}, v_i), where we take $v_0 = v_k$.

We say that the face is `strictly convex` if in addition we have that

1. no edge has zero length,
2. the angle at each vertex lies strictly between 0 and π, and
3. the face has non-zero area.

(note that the third is implied by the first and second).

For a vertex u, let $v_1, \ldots, v_k$ denote its neighbors in order. We can then consider the angles in the embedding between v_{i-1} and v_i at u. We denote these angles $\alpha_1, \ldots, \alpha_k$. As u is a convex combination of its neighbors, we have $\sum_i \alpha_i \geq 2\pi$. We say that u is a *wheel* if $\sum_i \alpha_i = 2\pi$. We say that u is a *strict wheel* if in addition each α_i is strictly between 0 and π.

We will make use of the following proposition, whose proof I leave as an exercise.

Proposition 21.2.1. *If (G, x, y) is a Tutte embedding of a planar graph in which every internal face is strictly convex and every internal vertex is a strict wheel, then it is a planar straight-line embedding of G.*

Our proof that Tutte embeddings are planar will follow from the following two theorems.

Theorem 21.2.2. *In a Tutte embedding of a planar graph, every face is convex and every vertex is a wheel.*

Theorem 21.2.3. *In a Tutte embedding of a three-connected planar graph, there are no edges of zero length, no angles of 0 or π and no faces of zero volume.*

21.3 Potentials

We will begin by establishing the notation we need to prove Theorem 21.2.2. Our first step will be to define a *potential* to be a map from the vertex set of the graph to the reals: $\sigma : V \to \mathbb{R}$. We say that a potential is *valid* if for all edges (u, v), $\sigma(u) \neq \sigma(v)$.

We say that two edges (u, v_0) and (u, v_1) are a *corner* if u, v_0 and v_1 all lie on some face, with u between v_0 and v_1 in the orientation of the face.

Given a potential f, we define a *change* to be a corner (u, v_0), (u, v_1) such that

1. some face contains both edges with orientation (v_0, u) and (u, v_1), and
2. $\text{sign}(\sigma(u) - \sigma(v_0)) \neq \text{sign}(\sigma(u) - \sigma(v_1))$.

Note that a change can be associated with both the face and the vertex u.

For each vertex u, we define

$$\text{changes}(u) = \text{the number of changes at } u$$

and for each face f, we define

$$\text{changes}(f) = \text{the number of changes at } f.$$

Note that to counte the changes at f we must be careful to consider the orientations.

Proposition 21.3.1. *For every valid potential,*

$$\sum_u \text{changes}(u) = \sum_f \text{changes}(f).$$

At faces, we will be more interested in the number of non-changes, defined by

$$\text{non} - \text{changes}(f) = \text{the number of sides of } f \ - \text{changes}(f).$$

Proposition 21.3.2. *For every valid potential on a planar graph*

$$\sum_u \text{changes}(u) + \sum_f \text{non} - \text{changes}(f) = 2(V-2) + F,$$

where V is the number of vertices and F is the number of faces in the graph.

Proof. We have

$$\begin{aligned}\sum_u \text{changes}(u) + \sum_f \text{non} - \text{changes}(f) &= \sum_u \text{changes}(u) + \sum_f (\text{sides}(f) - \text{changes}(f)) \\ &= \sum_f \text{sides}(f) \\ &= 2E,\end{aligned}$$

where E is the number of edges in the graph. The proposition now follows from Euler's Theorem, which says $V - E + F = 2$. □

Given a potential, we will say that a vertex u is *bracketed* by its neighbors if it has some neighbors v_i and v_j for which $\sigma(v_i) \leq \sigma(u) \leq \sigma(v_j)$.

The key to the proof of Theorem 21.2.2 is:

Lemma 21.3.3. *Let σ be a valid potential on a planar graph in which there are unique vertices of maximum and minimum potential, and every other vertex is bracketed by its neighbors. Then,*

1. *For every non-extreme vertex u,* $\text{changes}(u) = 2$. *and*
2. *For every face f,* $\text{non} - \text{changes}(f) = 2$.

Proof. As each of the non-extreme vertices are bracketed by their neighbors, each of them must have at least two changes.

Similarly, for each face, each local minimum and maximum of the potential must contribute a non-change. So, each face must have at least two non-changes.

Now, if each non-extreme vertex has the minimal possible number of changes, 2, each extreme vertex has no changes, and each face has the minimal possible number of non-changes, 2, then we get

$$\sum_u \text{changes}(u) + \sum_f \text{non} - \text{changes}(f) = 2(V-2) + F.$$

As this agrees with Proposition 21.3.2, each of these values must be at their minimum. □

For all but a finite number of lines through the origin, the projection of a Tutte embedding onto that line is a potential satisfying the conditions of Lemma 21.3.3 .

21.4 Proof of Theorem 21.2.2

Let (G, x, y) be a Tutte embedding, and let f be a face. If f is not convex, then there is a continuous family of lines that cut f in at least 4 edges. Thus, we can find a line that cuts f in at least 4 edges such that the projection of the embedding onto f is a valid potential. Moreover, because each vertex in a Tutte embedding is a convex combination of its neighbors, each non-extreme vertex is bracketed by its neighbors. However, between each consecutive pair of edges that are cut by the line, f must contain a local maximum or minimum. So, f must contain at least 4 non-changes, contradicting Lemma 21.3.3.

Similarly, if u is not a wheel, then there must be some line through u intersecting at least four wedges of the form (v_{i-1}, u, v_i) such that the projection of the embedding onto u is valid and bracketed. However, each of these wedges will contribute a change, contradicting Lemma 21.3.3.

21.5 Proof of Theorem 21.2.3

Given a potential σ, we will call an edge (u, v) *degenerate* if $\sigma(u) = \sigma(v)$. We will call a corner (u, v_0), (u, v_1) *degenerate* if both its edges are degenerate. Finally, we will call a vertex *degenerate* if each of its edges is degenerate.

In this section, we will prove

Theorem 21.5.1. *Let (G, x, y) be a Tutte embedding of a 3-connected planar graph, and let σ be a potential obtained by projecting the embedding onto a line. Then, G has no degenerate corner under σ.*

Before proving this theorem, let's observe that it implies Theorem B.

First, if there is an angle of 0 or π at a corner, (u, v_0), (u, v_1), then by projecting onto a line orthogonal to (u, v_0), we obtain a potential under which this corner is degenerate.

To see that is implies there is no edge of zero length, let (u, v_0) be such an edge, and let (u, v_1) be another edge that together with the first forms a corner. Now, project the embedding onto a line perpendicular to the line connecting u to v_1. Then, the corner will be degenerate under this embedding.

To prove Theorem 21.5.1, we will first prove that a degerate corner can only happen at a degernerate vertex.

Lemma 21.5.2. *Let (G, x, y) be a Tutte embedding of a 3-connected planar graph, and let σ be a potential obtained by projecting the embedding onto a line. If (u, v_0), (u, v_1) is a degenerate corner, then u must be degenerate.*

Proof. We first note that if u were non-degenerate, then it would have some neighbors v_i and v_j such that $v_i < u < v_j$.

Let s be one of the vertices at which σ is maximized and let t be one of the vertices at which σ is minimized. Note that both of these must lie on boundary face.

As the graph is three connected, there are either

- vertex-disjoint paths from v_0 to s and v_1 to t that do not go through u, or
- vertex-disjoint paths from v_0 to t and v_1 to s that do not go through u.

Let's assume without loss of generality that we are in the first case. Then, there is a simple path in the graph from t to v_1 to u to v_0 to s. It is not too difficult to show that there exists a valid bracketed potential τ such that

- $\tau(w) \neq \tau(v)$ for all vertices $w \neq v$,
- $\tau(t) = 1$, $\tau(s) = -1$, and
- τ is monotone strictly decreasing on this path.

By now considering the potentials $\sigma + \epsilon\tau$ and $\sigma - \epsilon\tau$, for sufficiently small ϵ, we can show that

1. for one of these potentials, u has at least 4 changes, and
2. for sufficiently small ϵ, both of these potentials are bracketed.

Thus, we obtain a contradiction to Lemma 21.3.3. □

Finally, we observe that Lemma 21.3.3 can be used to show that if there is any degenerate corner, then every interior vertex is degenerate. To prove this, consider a degenerate vertex u, and two of its neighbors v_0 and v_1. If there is no edge between v_0 and v_1, add one. The graph remains planar, 3-connected, and the embedding remains a Tutte embedding since v_0 and v_1 mapped to the same point anyway. Now, v_0 has a degenerate corner (v_0, u), (v_0, v_1), so v_0 must be degernate, and so on.