Specral Graph Theory and its Applications September 7, 2004

Lecture 2

Lecturer: Daniel A. Spielman

2.1 The Laplacian, again

We'll begin by redefining the Laplacian. First, let $G_{1,2}$ be the graph on two vertices with one edge. We define

$$L_{G_{1,2}} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & -1 \\ -1 & 1. \end{bmatrix}$$

Note that

$$x^T L_{G_{1,2}} x = (x_1 - x_2)^2. \tag{2.1}$$

In general, for the graph with n vertices and just one edge between vertices u and v, we can define the Laplacian similarly. For concreteness, I'll call the graph $G_{u,v}$ and define it by

$$L_{G_{u,v}}(i,j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = j \text{ and } i \in u, v \\ -1 & \text{if } i = u \text{ and } j = v, \text{ or } \textit{vice versa}, \\ 0 & \text{otherwise.} \end{cases}$$

For a graph G with edge set E, we now define

$$L(G) \stackrel{\text{def}}{=} \sum_{(u,v)\in E} L(G_{u,v}).$$

Many elementary properties of the Laplacian follow from this definition. In particular, we see that $L_{G_{1,2}}$ has eigenvalues 0 and 2, and so is positive semidefinite, where we recall that a symmetric matrix M is positive semidefinite if all of its eigenvalues are non-negative. Also recall that this is equivalent to

$$x^T M x \geq 0, \text{ for all } x \in R^n.$$

It follows immediately that the Laplacian of every graph is positive semidefinite. One way to see this is to sum equation (2.1) to get

$$x^T L_G x = \sum_{(u,v)\in E} (x_u - x_v)^2.$$

Remark Since the vertex set really doesn't matter, I actually prefer the notation $L(E)$ where E is a set of edges. Had I used this notation above, it would have eliminated some subscripts. For example, I could have written $L_{\{u,v\}}$ instead of $L_{G_{u,v}}$.

We now sketch the proof of one of the most fundamental facts about Laplacians.

Lemma 2.1.1. *Let $G = (V, E)$ be a connected graph, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of its Laplacian. Then, $\lambda_2 > 0$.*

Proof. Let x be an eigenvector L_G of eigenvalue 0. Then, we have

$$L_G x = \mathbf{0},$$

and so

$$x^T L_G x = \sum_{(u,v) \in E} (x_u - x_v)^2 = 0.$$

Thus, for each pair of vertices (u, v) connected by an edge, we have $x_u = x_v$. As the graph is connected, we have $x_u = x_v$ for all pairs of vertices u, v, which implies that x is some constant times the all 1s vector. Thus, the eigenspace of 0 has dimension 1. □

This lemma led Fiedler to consider the magnitude of λ_2 as a measure of how well-connected the graph is. In later lectures, we will see just how good a measure it is. Accordingly, we often call λ_2 the *Fiedler value* of a graph, and v_2 its *Fiedler vector*.

In general, it is pointless to consider unconnected graphs, because their spectra are just the union of the spectra of their components. For example, we have:

Corollary 2.1.2. *Let $G = (V, E)$ be a graph, and let L be its Laplacian. Then, the multiplicity of 0 as an eigenvalue of L equals the number of connected components of G.*

2.2 Some Fundamental Graphs

We now examine the eigenvalues and eigenvectors of the Laplacians of some fundamental graphs. In particular, we will examine

- The complete graph on n vertices, K_n, which has edge set $\{(u, v) : u \neq v\}$.
- The path graph on n vertices, P_n, which has edge set $\{(u, u + 1) : 1 \leq u < n\}$.
- The ring graph on n vertices, R_n, which has all the edges of the path graph, plus the edge $(1, n)$.
- The grid graph on n^2 vertices. To define this graph, we will identify vertices with pairs (u_1, u_2) where $1 \leq u_1 \leq n$ and $1 \leq u_2 \leq n$. Each node (u_1, u_2) has edges to each node that differs by one in just one coordinate.

Lemma 2.2.1. *The Laplacian of K_n has eigenvalue 0 with multiplicity 1 and n with multiplicity $n - 1$.*

Proof. Let L be the Laplacian of K_n. Let x be any vector orthogonal to the all 1s vector. Consider the first coordinate of Lx. It will be $n-1$ times x_1, minus

$$\sum_{u>1} x_u = -x_1,$$

as x is orthogonal to $\mathbf{1}$. Thus, x is an eigenvector of eigenvalue n. $\square$

Lemma 2.2.2. *The Laplacian of R_n has eigenvectors*

$$x_k(u) = \sin(2\pi k u/n), \text{ and}$$
$$y_k(u) = \cos(2\pi k u/n),$$

for $k \leq n/2$. Both of these have eigenvalue $2 - 2\cos(2\pi k/n)$. Note x_0 should be ignored, and y_0 is the all 1s vector. If n is even, then $x_{n/2}$ should also be ignored.

Proof. The best way to see that x_k and y_k are eigenvectors is to plot the graph on the circle using these vectors as coordinates. That they are eigenvectors is geometrically obvious. To compute the eigenvalue, just consider vertex $u = n$. $\square$

Lemma 2.2.3. *The Laplacian of P_n has the same eigenvalues as R_{2n}, and eigenvectors*

$$x_k(u) = \sin(\pi k u/n + \pi/2n).$$

for $0 \leq k < n$

Proof. This is our first interesting example. We derive the eigenvectors and eigenvalues by treating P_n as a quotient of R_{2n}: we will identify vertex u of P_n with both vertices u and $2n-u$ of R_{2n}. Let z be an eigenvector of R_{2n} in which $z(u) = z(2n-u)$ for all u. I then claim that the first n components of z give an eigenvector of P_n. To see why this is true, note that an eigenvector z of P_n must satisfy for some λ

$$2z(u) - z(u-1) - z(u+1) = \lambda z(u), \text{ for } 1 < u < n, \text{ and}$$
$$z(1) - z(2) = \lambda z(2)$$
$$z(n) - z(n-1) = \lambda z(n).$$

One can now immediately verify that the first n components of such a z satisfy these conditions. The only trickly parts are at the endpoints.

Now, to obtain such a vector z, we take

$$z_k(u) = \sin\left(2\pi k u/(2n) + \pi/(2n)\right),$$

which is in the span of x_k and y_k. $\square$

The quotient construction used in this proof is an example of a generally applicable technique.

2.2.1 Products of Graphs

We will now determine the eigenvalues and eigenvectors of grid graphs. We will go through a more general theory in which we relate the spectra of products graphs to the spectra of the graphs of which they are products.

Definition 2.2.4. *Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then $G \times H$ is the graph with vertex set $V \times W$ and edge set*

$$\Big((v_1, w), (v_2, w)\Big) \text{ where } (v_1, v_2) \in E \text{ and}$$
$$\Big((v, w_1), (v, w_2)\Big) \text{ where } (w_1, w_2) \in F.$$

Theorem 2.2.5. *Let $G = (V, E)$ and $H = (W, F)$ be graphs with Laplacian eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, and eigenvectors $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, respectively. Then, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, $G \times H$ has an eigenvector z of eigenvalue $\lambda_i + \mu_j$ such that*

$$z(v, w) = x_i(v) y_j(w).$$

Proof. To see that this eigenvector has the propper eigenvalue, let L denote the Laplacian of $G \times H$, d_v the degree of node v in G, and e_w the degree of node w in H. We can then verify that

$$\begin{aligned}
(Lz)(v, w) &= (d_v + e_w)(x_i(v) y_j(w)) - \sum_{(v,v_2) \in E} (x_i(v_2) y_j(w)) - \sum_{(w,w_2) \in F} (x_i(v) y_j(w_2)) \\
&= (d_v)(x_i(v) y_j(w)) - \sum_{(v,v_2) \in E} (x_i(v_2) y_j(w)) + (e_w)(x_i(v) y_j(w)) - \sum_{(w,w_2) \in F} (x_i(v) y_j(w_2)) \\
&= y_j(w) \left(d_v x_i(v) - \sum_{(v,v_2) \in E} (x_i(v_2)) \right) + x_i(v) \left(e_w y_j(w) - \sum_{(w,w_2) \in F} (x_i(v) y_j) \right) \\
&= y_j(w) \lambda_i x_i)(v) + x_i(v) \mu_j y_j(w) \\
&= (\lambda_i + \mu_j)(x_i(v) y_j(w)).
\end{aligned}$$

□

Thus, the eigenvalues and eigenvectors of the grid graph are completely determined by the eigenvalues and eigenvectors of the path graph. This completely explains the nice spectral embedding of the n-by-n grid graph: its smallest non-zero eigenvalue has multiplicity two and its eigenspace is spanned by the vectors $\mathbf{1} \times v_2$ and $v_2 \times \mathbf{1}$.

2.3 Bounding Eigenvalues

It is rare to be able to precisely determine the eigenvalues of a graph by purely combinatorial arguments. Usually, such arguments only provide bounds on the eigenvalues. The most commonly used tool for bounding eigenvalues is the Courant-Fischer characterization of eigenvalues:

Theorem 2.3.1 (Courant-Fischer). *Let A be an n-by-n symmetric matrix and let $1 \leq k \leq n$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then,*

$$\lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T A x}{x^T x}, \tag{2.2}$$

and

$$\lambda_k = \max_{S \text{ of dim } n-k-1} \min_{x \in S} \frac{x^T A x}{x^T x}, \tag{2.3}$$

For example, this allows us to obtain an easy upper bound on $\lambda_2(P_n)$. Setting $S = \{x : x \perp \mathbf{1}\}$, and applying (2.3), we obtain

$$\lambda_2(L) \leq \min_{x \perp \mathbf{1}} \frac{x^T A x}{x^T x}.$$

This allows us to obtain an easy upper bound on $\lambda_2(P_n)$. Consider the vector x such that $x(u) = (n+1) - 2u$. This vector satisfies $x \perp \mathbf{1}$, so

$$\begin{aligned} \lambda_2(P_n) &\leq \frac{\sum_{1 \leq u < n} (x(u) - x(u+1))^2}{\sum_u x(u)^2} \\ &= \frac{\sum_{1 \leq u < n} 2^2}{\sum_i (n+1-2u)^2} \\ &\sim \frac{4n}{n(n^2/3)} \\ &= \frac{12}{n^2}. \end{aligned}$$

So, we can easily obtain an upper bound on $\lambda(P_n)$ that is of the right order of magnitude.

It is more difficult to obtain lower bounds. In the next lecture, we will introduce a technique for proving lower bounds.

We will apply the technique to the example of the complete binary tree. The complete binary tree on $n = 2^d - 1$ nodes, B_n, is the graph with edges of the form $(u, 2u)$ and $(u, 2u+1)$ for $u < n/2$. To upper bound $\lambda_2(B_n)$, we construct a vector x as follows. We first set $x(1) = 0$, $x(2) = 1$, and $x(3) = -1$. Then, for every vertex u that we can reach from node 2 without going through node 1, we set $x(u) = 1$. For all the other nodes, we set $x(u) = -1$. We then have

$$\frac{\sum_{(u,v) \in B_n} (x_u - x_v)^2}{\sum_u x_u^2} = \frac{(x_1 - x_2)^2 + (x_1 - x_3)^2}{n} = 2/n.$$

For now, we leave with some lower bounds on λ_n.

Lemma 2.3.2. *Let $G = (V, E)$ be a graph and let vertex $w \in V$ have degree d. Then,*

$$\lambda_{max}(G) \geq d.$$

Proof. By (2.2),

$$\lambda_{max}(G) \geq \max_x \frac{x^T L(G) x}{x^T x}.$$

Let x be the vector given by

$$x(u) = \begin{cases} 1 & u = w \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\frac{\sum_{(u,v)\in E}(x_u - x_v)^2}{\sum x_u^2} = \frac{d}{1} = d.$$

$\square$

We can improve this lemma slightly.

Lemma 2.3.3. *Let $G = (V, E)$ be a graph and let vertex $w \in V$ have degree d. Then,*

$$\lambda_{max}(G) \geq d + 1.$$

Proof. Let x be the vector given by

$$x(u) = \begin{cases} d & u = w \\ -1 & \text{if } (u, w) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\frac{\sum_{(u,v)\in E}(x_u - x_v)^2}{\sum x_u^2} = \frac{d(d+1)^2}{d^2 + d} = d + 1.$$

$\square$

Can we prove a larger lower bound? To find out, let's consider the star graph with n vertices, S_n. It has edges of the form $(1, i)$ for $2 \leq i \leq n$. To determine its eigenvalues, we apply the following lemma

Lemma 2.3.4. *Let $G = (V, E)$ be a graph, and let v_1 and v_2 be vertices of degree one that are both connected to another vertex w. Then, the vector x given by*

$$x(u) = \begin{cases} 1 & u = v_1 \\ -1 & u = v_2 \\ 0 & \textit{otherwise,} \end{cases}$$

is an eigenvector of the Laplacian of G of eigenvalue 1.

Proof. One can immediately verify that $Lx = x$. $\square$

Thus, we see that S_n has eigenvectors of eigenvalue 1 pairing vertices 2 and 3, 3 and 4, up to 2 and n. The space spanned by these vectors has dimension $n-2$. As the vector $\mathbf{1}$ is also an eigenvector, only one vector may remain. We don't even have to compute it to determine its eigenvalue: the sum of the eigenvalues is the trace of the matrix, which is $2(n-1)$. The $n-1$ eigenvectors we have found so far account for $n-2$ of this sum, so the last eigenvector must have eigenvalue n. This establishes that Lemma 2.3.4 is tight. In fact, this last eigenvector turns out to be the vector constructed in the proof of that lemma.