Preconditioning, I

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19.1 Appologies

Problem 3a on Problem Set 2 was also false! I am very very very sorry. To make sure that this never happens again, I am cancelling all future problem sets in this class.

However, if someone really wants another problem set, I may be willing to make an exception.

19.2 No Class Tuesday: Seminar Instead

I will be giving an Applied Math Seminar on Tuesday at 4:15. Accordingly, we will not hold class. I'll be talking about preconditioning in the seminar, and it will be accessible to a general audience. That means that you will understand more than most. The talk will tie together many of the things that we've covered in this class, and I recommend it highly.

19.3 Preconditioning

Recall the notation from lecture 3:

$$L_A \preccurlyeq L_B$$

if $L_B - L_A$ is positive semi-definite, which is equivalent to

$$x^T L_A x \geq x^T L_B x,$$

for all $x \in \mathbb{R}^n$.

Our goal for today is to prove the following theorem:

Theorem 19.3.1. *Let A be an unweighted graph on n vertices. For every $0 \leq k < n$, there exists a subgraph B of A with $n - 1 + k^2$ edges such that*

$$L_B \preccurlyeq L_A O((\log n \log\log n)^2)\frac{m}{k+1} \preccurlyeq B.$$

We'll begin by considering the case in which $k = 0$, in which case B is a spanning tree.

But first, let's observe that the left-hand inequality is a triviality.

Lemma 19.3.2. *If B is a subgraph of A, then*

$$L_B \preccurlyeq L_A.$$

Proof. Let C be the graph containing all edges of A that do not appear in B. Then,

$$L_A - L_B = L_C,$$

which is positive semi-definite because it is a Laplacian matrix. □

19.4 Upper bound technique

We will now introduce a technique for proving upper bounds like $L_A \preccurlyeq \alpha L_B$. In fact, it will be the same technique that we used in Lecture 3 to lower bound λ_2. But, since that was a long time ago, let me refresh your memories.

We proved:

Lemma 19.4.1. *Let $P_{u,v}$ be a path between vertex u and vertex v of length k. Then,*

$$L_{u,v} \preccurlyeq k L_{P_{u,v}}.$$

We will use this to prove:

Lemma 19.4.2. *Let $G = (V, E)$ and $H = (V, F)$, where $F \subseteq E$. For each edge $(u, v) \in E$, let $P_{u,v}$ be a path in F from u to v. Then,*

$$L_G \preccurlyeq \alpha L_H,$$

where

$$\alpha = \max_{(a,b)\in F} \sum_{(u,v)\in E:(a,b)\in P_{u,v}} \text{length}(P_{u,v}).$$

Proof. We write G as a sum of the Laplacians of its edges, and then support each with the corresponding path in H:

$$\begin{aligned} L_G &= \sum_{(u,v)\in E} L_{u,v} \\ &\leq \sum_{(u,v)\in E} \text{length}(P_{u,v}) L_{P_{u,v}}. \end{aligned}$$

Now, the total weight on edge $(a, b) \in F$ is

$$\sum_{(u,v)\in E:(a,b)\in P_{u,v}} \text{length}(P_{u,v}) L_{P_{u,v}},$$

and so

$$\sum_{(u,v)\in E} \text{length}(P_{u,v}) L_{P_{u,v}} \preccurlyeq \left(\max_{(a,b)\in F} \sum_{(u,v)\in E:(a,b)\in P_{u,v}} \text{length}(P_{u,v}) \right) L_F.$$

□

Let's do an example. Consider the s-by-s grid graph, with $n = s^2$ nodes. We'll precondition it by the tree shown in Figure ?.

I'll make the following conjecture.

Conjecture 19.4.3. *There exists an absolute constant α such that every connected graph G with m edges on n vertices contains a subgraph H with $n-1$ edges such that*

$$L_H \preccurlyeq L_G \preccurlyeq \alpha m \cdot L_H.$$

We'll miss proving this conjecture by a factor of $(\log n \log\log n)^2$.

We will use the following theorem of Elkin, Spielman and Teng:

Theorem 19.4.4. *Every connected graph $G = (V, E)$ contains a spanning tree T so that if $P_{u,v}$ denotes the unique path in T between u and v, then*

$$\sum_{(u,v)\in E} \mathrm{length}(P_{u,v}) \leq O(m(\log n \log\log n)^2).$$

By Lemma 19.4.2, this immediately implies that

$$L_G \preccurlyeq O(m(\log n \log\log n)^2) L_T.$$

19.5 Trees

Now, let's consider how well one can precondition a graph by a subtree.