Towards a Type Theory of Contexts

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Invited Talk
Workshop on Mechanized Reasoning about Languages with Variable Binding (Merλ in’05)
Tallinn, Estonia, September 30, 2005

Joint work with Aleks Nanevski and Brigitte Pientka

Work in progress!
Context

• Dictionary definition (Merriam-Webster Online)
  a. *the parts of a discourse that surround a word or passage and can throw light on its meaning*
  b. *the interrelated conditions in which something exists or occurs*

• Of central importance in computer science
  • Computational Linguistics
  • Artificial Intelligence
  • Programming Languages

• Not very well understood
Logic and Type Theory

- *Logic* in this talk always means *intuitionistic logic* and is therefore immediately computational
  - Logical frameworks
  - Functional programming
- *Type theory* makes proof terms explicit
  - Canonical forms
  - Evaluation
Outline

- Goals and Methods
- Validity
- Contextual Validity
- Context and Substitution Variables
- Conclusion
Original Goals

- Understanding *meta-variables*
  - Logical essence
  - Simple and dependent types
  - Reflection?

- In logical frameworks
  - Unification
  - Proof search (subgoals)

- For staged computation
  - Manipulating “open code”
  - Filling holes
Spin-offs

• Understanding *explicit substitutions*
  • Logical foundation
  • Substitution variables, quantification
  • Sequent calculi?

• Understanding *contexts*
  • Logical meaning
  • Context variables
  • Context quantification (?)
Methodology

• Separating judgments from propositions [Martin-Löf’83]

• Categorical judgments [Pf.&Davies’01]

• Other applications of modal type theory
  • Monads [Moggi’88,’91][Pf.&Davies’01]
  • Run-time code generation [Davies&Pf.’96,’01]
  • Partial evaluation [Davies’96]
  • Distributed computation [Murphy,Crary,Harper,Pf.’04]
Some Consequences and Criteria

• Respecting $\alpha$-conversion
• Computation arises from reduction
• Normalization
• Canonical forms
• Orthogonality of language constructs
  • Adding other propositions, types
  • Modular reasoning
Hypothetical Judgments

- Basic judgment $A \text{ true}$
- Hypothetical judgment

$$A_1 \text{ true}, \ldots, A_n \text{ true} \vdash A \text{ true}$$

- Hypothesis rule

$$A \text{ true} \in \Gamma$$

$$\Gamma \vdash A \text{ true}$$

- Substitution principle

If $\Gamma \vdash A \text{ true}$ and $\Gamma, A \text{ true} \vdash C \text{ true}$ then $\Gamma \vdash C \text{ true}$
Modal Logic of Validity

- Categorical judgment

\[ \bullet \vdash A \text{ true} \quad A \text{ valid} \]

- Generalized hypothetical judgment

\[ \Delta; \cdots; A_n \text{ true} \vdash B_1 \text{ valid}, \ldots, B_m \text{ valid}; C \text{ true} \]

- Generalized definition of validity

\[ \Delta; \bullet \vdash A \text{ true} \quad \Delta; \Gamma \vdash A \text{ valid} \]
Modal Logic of Validity, Ctd

- New hypothesis rule

\[
\frac{A \text{ valid} \in \Delta}{\Delta; \Gamma \vdash A \text{ true}}
\]

- New substitution principle

If \( \Delta; \bullet \vdash A \text{ true} \) and \( \Delta, A \text{ valid}; \Gamma \vdash C \text{ true} \) then

\( \{\Delta; \Gamma \vdash C \text{ true}} \)
Simple Type Theory of Validity

• Assign proof terms

\[
\begin{align*}
\Delta; \Gamma \vdash M : C \\
u_1 :: B_1, \ldots, u_m :: B_m; x_1 : A_1, \ldots, x_n : A_n \vdash M : C
\end{align*}
\]

• Hypothesis rules

\[
\begin{align*}
\Delta; \Gamma \vdash x : A &\quad \frac{x : A \in \Gamma}{\Delta; \Gamma \vdash x : A} \\
\Delta; \Gamma \vdash u : A &\quad \frac{u :: A \in \Delta}{\Delta; \Gamma \vdash u : A}
\end{align*}
\]

• Ordinary substitution principle

If \( \Delta; \Gamma \vdash M : A \) and \( \Delta; \Gamma, x : A \vdash N : C \) then
\( \Delta; \Gamma \vdash [M/x]N : C \)
Substitution Operations

• Modal substitution principle
  If $\Delta; \bullet \vdash M : A$ and $\Delta, u::A; \Gamma \vdash N : C$ then
  $\Delta; \Gamma \vdash [M/u]N : C$

• Ordinary substitution $[M/x]N$ as usual

• Modal substitution $[M/u]N$ slightly unusual

  $[M/x](\lambda y. N) = \lambda y. [M/x]N$ for $y \not\in \text{FV}(M)$

  $[M/u](\lambda y. N) = \lambda y. [M/u]N$ without proviso

• $[M/u]$ can be implemented by “grafting”
Excursion: Higher-Order Unification

- Huet’s formulation
- Substituends for meta-variables
  - Have no free ordinary variables
  - May contain other meta-variables

\[
\lambda x. \lambda y. \lambda z. u_1 \ x \ y \ = \ \lambda x. \lambda y. \lambda z. u_2 \ y \ z
\]

\[
\begin{align*}
u_1 \ &\leftarrow \ (\lambda x. \lambda y. u_3 \ y) \\
u_2 \ &\leftarrow \ (\lambda y. \lambda z. u_3 \ y)
\end{align*}
\]

- Precisely modeled by validity
**Internalizing Validity**

- Validity is a categorical judgment, not type
- Internalize as modal type constructor
- By Curry-Howard, can be read as proposition

\[
\Delta; \bullet \vdash M : A \\
\Delta; \Gamma \vdash \text{box} \ M : \Box A \quad \Box I
\]

\[
\Delta; \Gamma \vdash M : \Box A \quad \Delta, u :: A; \Gamma \vdash N : C \\
\Delta; \Gamma \vdash \text{let box} \ u = M \ \text{in} \ N : C \quad \Box E
\]
Computation from Reduction

• Reduce when eliminations follow introductions

\[(\beta) \quad \text{let box } u = \text{box } M \text{ in } N \rightarrow [M/u]N\]

• Expand to create eliminations followed by introductions

\[(\bar{\eta}) \quad M : \Box A \rightarrow \text{let box } u = M \text{ in box } u\]

• Omit reduction, expansion for \(A \rightarrow B\)

• Reduction and expansion preserve types
  • Follows from substitution properties
Excursion: Staged Computation

- □A is *source code of type* A
- Implement as run-time code generation

\[
\begin{align*}
\text{exp} & : \text{nat} \rightarrow □(\text{nat} \rightarrow \text{nat}) \\
\exp 0 &= \text{box} (\lambda x. 1) \\
\exp 1 &= \text{box} (\lambda x. x) \\
\exp n &= \text{let box } u = \exp (n - 1) \text{ in box } (\lambda x. (u x) \ast x) \\
& \text{ for } n \geq 2 \\
\exp 2 &\mapsto^* \text{box} (\lambda x_2. (\lambda x_1 x_1) x_2 \ast x_2) \\
&\mapsto^* \text{box} (\lambda x_2. x_2 \ast x_2)
\end{align*}
\]
Characteristic Laws

- Laws of (intuitionistic) S4
  \[ \vdash \square A \rightarrow A \]
  \[ \vdash \square A \rightarrow \square \square A \]
  \[ \vdash \square (A \rightarrow B) \rightarrow \square A \rightarrow \square B \]
  \[ \not\vdash A \rightarrow \square A \]

- Also need necessitation (\(\approx \square I\))

- Kripke interpretation
  - \(\square A\) true means \(A\) true in all future worlds
  - Accessibility is reflexive and transitive
Contextual Validity

- $A \ valid[\Psi]$ means $A$ is true in every world where all hypothesis in context $\Psi$ are satisfied.

- Generalized hypothetical judgment

\[
\Delta; \Psi \vdash A \ true, \ldots, A_n \ true \vdash C \ true \\
\Delta; \Psi \vdash A \ true
\]

- Generalized definition of contextual validity

\[
\Delta; \Psi \vdash A \ valid[\Psi]
\]
Validity and Contextual Validity

- \( A \) valid[\( \bullet \)] corresponds to \( A \) valid
- \( A \) valid[\( A_1 \ true, \ldots, A_n \ true \)] like \( (A_1 \rightarrow \cdots \rightarrow A_n \rightarrow A) \) valid
  - Proof theory and computation is quite different
Proof Terms

• Assign proof terms

\[ u_1 :: B_1[\Psi_1], \ldots, u_m :: B_m[\Psi_m]; x_1:A_1, \ldots, x_n:A_n \vdash M : C \]

\( \Delta \)
\( \Gamma \)

• Contextual hypothesis rule

\[ u :: A[\Psi] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Psi \]

\[ \Delta; \Gamma \vdash u[\sigma] : A \]

• Requires explicit substitution \( \sigma \) to establish that all assumptions in \( \Psi \) can be realized in \( \Gamma \)
Explicit Substitutions

• Simultaneous substitutions

Substitutions $\sigma ::= \bullet | \sigma, M/x$

• Typing judgment $\Delta; \Gamma \vdash \sigma : \Psi$

\[
\Delta; \Gamma \vdash \bullet : \bullet \quad \frac{\Delta; \Gamma \vdash \sigma : \Psi}{\Delta; \Gamma \vdash (\sigma, M/x) : (\Psi, x:A)} \quad \frac{\Delta; \Gamma \vdash M : A}{\Delta; \Gamma \vdash (\sigma, M/x) : (\Psi, x:A)}
\]

• Not just a tuple (cf dependencies)
Substitution Principle

- Substitution principles for contextual validity
  \[ \text{If } \Delta; \Psi \vdash M : A \text{ and } \Delta, u :: A[\Psi]; \Gamma \vdash N : C \text{ then } \]
  \[ \Delta; \Gamma \vdash \sub{\Psi.M/u}{N} : C \]

- Also need to substitute into explicit substitutions!
  \[ \text{If } \Delta; \Psi \vdash M : A \text{ and } \Delta, u :: A[\Psi]; \Gamma \vdash \tau : \Phi \text{ then } \]
  \[ \Delta; \Gamma \vdash \sub{\Psi.M/u}{\tau} : \Phi \]

- Close \( M \) over \( \Psi \) to preserve \( \alpha \)-conversion
  - Not necessary in nameless implementation
Substitution Operations

- $[\Psi.M/u]N$ as before, except

  $$[\Psi.M/u](w[\sigma]) = w[[\Psi.M/u]\sigma] \quad \text{for } u \neq w$$

  $$[\Psi.M/u](u[\sigma]) = [\sigma'/\Psi](M) \quad \text{where } \sigma' = [\Psi.M/u]\sigma$$

- $\sigma'/\Psi$ renames variables in $\sigma'$ to match $\Psi$
- Need to awaken the postponed substitution $\sigma$
- Different occurrences of $u$ may be under different substitutions
Simultaneous Substitution

- Simultaneous substitution principles

  If $\Delta; \Psi \vdash \sigma : \Gamma$ and $\Delta; \Gamma \vdash M : A$ then $\Delta; \Psi \vdash [\sigma]M : A$

- Definition of $[\sigma]M$ is straightforward, for example

  $[\sigma](x) = M$ where $M/x \in \sigma$

  $[\sigma](u[\tau]) = u[[\sigma]\tau]$

  $[\sigma](\lambda x. M) = \lambda x. [\sigma, x/x]$

- Composition of substitutions (also straightforward)

  If $\Delta; \Psi \vdash \sigma : \Gamma$ and $\Delta; \Gamma \vdash \tau : \Phi$ then $\Delta; \Psi \vdash [\sigma]\tau : \Phi$
Excursion: Unification Revisited

- Meta-variables à la Dowek et al. in h.o. unification
- Substituends for meta-variables
  - Have free ordinary variables
  - Occur under explicit substitution

\[
\begin{align*}
  u_1 &:: i[x:i, y:i], u_2:: i[y:i, z:i] \\
  \lambda x. \lambda y. \lambda z. u_1[x/x, y/y] &\equiv \lambda x. \lambda y. \lambda z. u_2[y/y, z/z] \\
  u_1 &\leftarrow u_3[y/y] \\
  u_2 &\leftarrow u_3[y/y] \\
  \text{for } u_3:: i[y:i]
\end{align*}
\]
Internalizing Contextual Validity

• New contextual modal operator $[\Psi]A$

$$
\Delta; \Psi \vdash M : A \\
\Delta; \Gamma \vdash \text{box } (\Psi . M) : [\Psi]A
$$

$$
\Delta; \Gamma \vdash M : [\Psi]A \quad \Delta, u::A[\Psi]; \Gamma \vdash N : C \\
\Delta; \Gamma \vdash \text{let box } u = M \text{ in } N : C
$$

• Reduction

$$(\beta) \quad \text{let box } u = \text{box } (\Psi . M) \text{ in } N \rightarrow [(\Psi . M)/u]N$$
Identity Substitutions

- Expansion requires identity substitution

\[
(\bar{\eta}) \quad M : [\Psi]A \rightarrow \text{let } \text{box } u = M \text{ in box } (\Psi.u[\text{id}(\Psi)])
\]

- Define

\[
\text{id}(\bullet) = \bullet \\
\text{id}(\Psi, x:A) = \text{id}(\Psi), x/x
\]

- Reduction and expansion preserve types
Excursion: Staging Revisited

• Avoid creating redexes in code generation

• Manipulate open code

  \[ [\Psi]A \text{ — source with free variables in } \Psi \]

  \[
  \begin{align*}
  \text{exp} & : \text{nat} \rightarrow [x:\text{nat}]\text{nat} \\
  \text{exp} \ 0 & = \text{box} \ (x. \ 1) \\
  \text{exp} \ 1 & = \text{box} \ (x. \ x) \\
  \text{exp} \ n & = \ \text{let box } u = \text{exp} \ (n - 1) \ \text{in box} \ (x. (u[x/x]) * x) \\
  & \quad \text{for } n \geq 2 \\
  \text{exp} \ 2 & \leftrightarrow^* \text{box} \ (\lambda x_2. (\lambda x_1.x_1) \ x_2 * x_2) \\
  & \leftrightarrow^* \text{box} \ (x_2. x_2 * x_2)
  \end{align*}
  \]
Some Sample Theorems

\[ \vdash \bullet A \rightarrow A \]
\[ \vdash [x: B] A \rightarrow [y: C][x: B] A \]
\[ \vdash [x: C](A \rightarrow B) \rightarrow [x: C] A \rightarrow [x: C] B \]
\[ \vdash [x: A] A \]
\[ \vdash [x: B, y: B] A \rightarrow [z: B] A \]
\[ \vdash [x: B] A \rightarrow [x: B, y: C] A \]
\[ \not \vdash [x: B] A \rightarrow A \]
\[ \not \vdash A \rightarrow [x: B] A \]
Context Variables

- Theorems parametric in propositions \((A,B,C)\)

\[ \vdash [x:B]A \rightarrow [y:C][x:B]A \]

- Would like theorems parametric in contexts

- Write \(\psi, \phi, \gamma\) for context variables

\[ \vdash [\psi]A \rightarrow [\phi][\psi]A \]

\[ \lambda x:[\psi]A. \text{let box } u = x \text{ in box } (\phi. \text{box}\ (\psi. u[\text{id}_{\psi}])) \]
Identity Substitutions

- Extended language

  Contexts  \[ \Gamma ::= \bullet | \Gamma, x:A | \gamma \]

  Substitutions  \[ \sigma ::= \bullet | \sigma, M/x | \text{id}_\gamma \]

- Typing

  \[ \Delta; \gamma, \Gamma \vdash \text{id}_\gamma : \gamma \]

- Substituting for context variables

  - box \((\Psi. M)\) does not bind context variable at head of \(\Psi\)
  - \(\{\Gamma/\gamma\}(\text{id}_\gamma) = \text{id}(\Gamma)\) expands
Sample Theorems Revisited

\[ \vdash [\bullet]A \rightarrow A \]
\[ \vdash [\psi]A \rightarrow [\phi][\psi]A \]
\[ \vdash [\psi](A \rightarrow B) \rightarrow [\psi]A \rightarrow [\psi]B \]
\[ \vdash [x:A]A \]
\[ \vdash [\psi, x:B, y:B]A \rightarrow [\psi, z:B]A \]
\[ \vdash [\psi]A \rightarrow [\psi, x:B]A \]
Excursion: Closures

- Closures in functional languages, \( \text{clo}(\eta, \nu) \)
- Object language typing
  \[
  \frac{\vdash \eta : \Psi \quad \Psi \vdash \nu : \tau}{\vdash \text{clo}(\eta, \nu) : \tau}
  \]
- For direct representation, need context variables and internalized substitutions (here \( \psi[\bullet] \))

\[
\text{clo} : [\bullet]\psi \to [\psi]A \to [\bullet]A
\]
Substitution Meta-Variables

- Substitution variables are meta-variables
- New substitutions and types

Types

\[ A ::= a \mid A_1 \rightarrow A_2 \mid [\Psi]A \mid [\Psi]\Phi \]

Substitutions

\[ \sigma ::= \bullet \mid \sigma, M/x \mid \text{id}_\gamma \mid s[\sigma] \]

Meta-Contexts

\[ \Delta ::= \bullet \mid \Delta, u::A[\Psi] \mid \Delta, s::\Phi[\Psi] \mid \Delta, \gamma \text{ ctx} \]

- New hypothesis rule for substitutions

\[
\frac{s::\Phi[\Psi] \in \Delta}{\Delta; \Gamma \vdash \tau : \Psi} \vdash \Delta; \Gamma \vdash s[\tau] : \Phi
\]
Extended Term Language

- Proposal for new terms, to be revised

\[
\Delta; \Psi \vdash \sigma : \Phi \\
\hline
\Delta; \Gamma \vdash \text{sbox} (\Psi . \sigma) : [\Psi] \Phi
\]

\[
\Delta; \Gamma \vdash M : [\Psi] \Phi \\
\Delta, s::\Phi[\Psi]; \Gamma \vdash N : C
\]

\[
\Delta; \Gamma \vdash \text{let sbox} \ s = M \ \text{in} \ N : C
\]

- Substitution properties, reduction straightforward
- Example, internalizes substitution property

\[
\vdash \lambda x . \lambda y . \text{let sbox} \ s = x \ \text{in} \ \text{let box} \ u = y \ \text{in} \ \text{box} \ (\psi . u[s[\text{id}_\psi]])
\]

\[
: [\psi] \phi \rightarrow [\phi] A \rightarrow [\psi] A
\]
Destructing Substitutions

- Problem: there are no destructors for substitutions

\[[\bullet](\phi, x:A) \rightarrow [\bullet]\phi\]

\[\lambda y. \text{let } s = y \text{ in sbox} \]

- Solution 1: projections of substitutions \((\text{hd}, \text{tl}_x)\)

\[\lambda y. \text{let } s = y \text{ in sbox } (\text{hd}(s[\text{id}_\phi, x/x]))\]

- Solution 2: pattern matching for substitutions

\[\lambda y. \text{let } s_\phi, u_x = y \text{ in sbox } (s[\text{id}_\phi])\]
Pattern Matching Substitutions

• Tentatively adopt solution 2

\[ \Delta; \Gamma \vdash M : [\Psi]\Phi \quad \Delta' = \delta([\Psi]\Phi) \quad \Delta, \Delta'; \Gamma \vdash N : C \]

\[ \Delta; \Gamma \vdash \text{let sbox } \Delta' = M \text{ in } N : C \]

where (subject to renaming)

\[ \delta([\Psi]\bullet) = \bullet \]
\[ \delta([\Psi]\phi) = s_\phi \]
\[ \delta([\Psi]\Phi, x:A) = \delta([\Psi]\Phi), u_x::A[\Psi] \]

• Substitution for context variables expands patterns
Language Summary So Far

- Contextual modal types and simultaneous substitutions

Types \( A ::= a | A_1 \rightarrow A_2 | [\Psi]A | [\Psi]\Phi \)

- Context variables
- Substitution principles
- Type preservation for reduction, expansion
- Strong normalization by interpretation (?)
Further Considerations

• Dependent types
• Context quantification
• Contextual modality like $\diamond A (\langle \Psi \rangle A)$
• Context concatenation ($\gamma ctx[\Psi]$)
Dependent Types

- System engineered to permit dependent types
  - Functional version (only sketched)
  - Logical frameworks version [Nanevski, Pf., Pientka’05]
  - No context variables, internal substitutions so far
- Contexts and substitutions are dependent

\[
\begin{align*}
\Delta \vdash \Gamma & \quad \Delta; \Gamma \vdash A : type \\
\Delta \vdash \Gamma, x:A & \quad \Delta; \Gamma \vdash M : [\sigma]A \\
\Delta \vdash \Gamma & \quad \Delta; \Gamma \vdash (\sigma, M/x) : (\Psi, x:A)
\end{align*}
\]

- Canonical forms via \textit{hereditary substitutions}
  [Watkins, Cervesato, Pf., Walker’02]

Merlin’05, Tallinn, Sep’05 – p.41
Staged computation with “open code”
Dependent types for reasoning about staged programs
  Modalities for correct staging
  Dependent types for functional correctness
Hypothetical example

\[ \exp : \prod n : \text{nat}. \Sigma r : [x : \text{nat}] \text{nat}. \]
\[ \prod k : \text{nat}. \text{let} \text{box } r = u \text{ in } u[k/x] \doteq k^n \]

Not “field-tested”
Apps: Logical Frameworks

- Model meta-variables
  - For unification
  - For proof search
- Efficient implementation via deBruijn indexes
- Logical foundation for Twelf implementation (almost)
  - Wish I had time to rewrite unification
Context Quantification

- Speculative
- $\forall \gamma. A$ is highly impredicative
  - May be rejected on philosophical grounds
  - Predicative form (?)
- Inductively defined regular worlds [Schürmann’00]
  - Example
    \[
    W ::= \bullet \mid W, x:exp \mid W, t:tp
    \]
    \[
    \forall \gamma \in W. A
    \]
  - Relevant to Delphin [Schürmann’02]?
Selected Other Related Work

- Philosophy, artificial intelligence [. . . many. . . ]
  - Generally classical logic
  - No computational interpretation
- Functional programming or staged computation
  - $\lambda\kappa\epsilon$-calculus [Sato, Sakurai, Kameyama’02]
  - Environment classifiers [Taha & Nielson’03]
- Logical frameworks or theorem proving
  - Meta-variables with contexts [Dowek, Hardin, Kirchner’95]
  - Dependent typing in deBruijn form [Muñoz’01]
  - Axiomatic approach [Honsell, Miculan, Scagnetto’01]
  - Meta$^n$-variables [Sato, Sakurai, Kameyama, Igarashi’03]
Conclusion

• Goal: understanding meta-variables
• Developed contextual modal logic
  • Explicit substitutions inevitable
  • Simply and dependently typed versions
• Staged computation
  • Manipulating open code
  • Reasoning about staged programs
• Logical frameworks
  • Meta-variables for unification, search
  • Basis for efficient implementation
More Information

*Contextual Modal Type Theory*
Aleksandar Nanevski, Frank Pfenning, and Brigitte Pientka
Submitted, September 2005.
http://www.cs.cmu.edu/~fp/papers/cmtt05.pdf