15-815 Automated Theorem Proving
Frank Pfenning, Spring 2004
Out Mar 3, due March 18
80 pts

Assignment 5: Quantification

In this assignment we explore the laws of quantification in first-order logic and logical translations.

Question 1 (10 pts): Classical First-Order Logic

Extend the sequent calculus for the judgment $\Gamma \# \Delta$ to allow universal and existential quantification.

Question 2 (30 pts): First-Order Sequent Calculus

For each of the following sequents $\Gamma \Longrightarrow C$, give a proof in intuitionistic sequent calculus if it exists. If not, give a proof in classical sequent calculus for $\Gamma \# C$, if it exists, interpreting all the connectives and quantifiers classically. If neither exists, just say so. In problems 5 and 6 you should assume that x does not occur free in A.

1. $\neg\forall x.\ A(x) \Longrightarrow \exists x.\ \neg A(x)$
2. $\exists x.\ \neg A(x) \Longrightarrow \neg\forall x.\ A(x)$
3. $\neg\exists x.\ A(x) \Longrightarrow \forall x.\ \neg A(x)$
4. $\forall x.\ \neg A(x) \Longrightarrow \neg\exists x.\ A(x)$
5. $A \supset (\exists x.\ B(x)) \Longrightarrow \exists x.\ (A \supset B(x))$
6. $\exists x.\ (A \supset B(x)) \Longrightarrow A \supset (\exists x.\ B(x))$
7. $\cdot \Longrightarrow \exists x.\ \forall y.\ (P(x) \supset P(y))$

Question 3 (20 pts): Double-Negation Translation

Extend the double-negation translation $()^o$ from classical to intuitionistic logic to account for universal and existential quantification. Show the new cases for existential quantification in the proof of Lemma 3.18 of the notes:

If $\Gamma \# \Delta$ *then* $\Gamma^o, \neg\Delta^o \Longrightarrow p$ *for a parameter* p *not in* Γ *or* Δ*.*

Question 4 (20 pts): Implicational Fragment

In classical propositional logic, we can put any formula into a conjunctive normal form (CNF) in order to simplify the implementation of SAT solvers. A *literal* is an atom P or a negated atom $\sim P$. A *clause* is a disjunction of literals, where the empty disjunction represents falsehood. A *conjunctive normal form* is a conjunction of clauses, where the empty conjunction represents truth. This normal form can be extended to the first-order case by Skolemization.

An exact analogue of CNF does not exist in intuitionistic logic. However, we can translate any proposition A into the *implicational fragment* so that A is provable iff its translation is provable. Give such a translation from full propositional intuitionistic logic which includes implication, conjunction, truth, disjunction, and falsehood. Your translation may introduce new propositional parameters as needed.

If possible, your translation should be such that it extends to the first-order case where the target fragment contains only implication and universal quantification.

You do not need to prove the correctness of your translation.