Lecture Notes on Resource Semantics

15-816: Modal Logic
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1 Introduction

In this lecture we continue our investigation of the resource semantics for linear logi from the previous lecture. We first consider the so-called additive connectives of linear logic. While we will be able to develop a resource sequent calculus which is still in bijective correspondence with the linear sequent calculus, it suggests a more elementary resource semantics which also has an elegant natural deduction formulation. This can be accomplished by "untethering" the left rules of the resource sequent calculus as much as possible.

The material in this lecture is based on work by Jason Reed [Ree07, Ree09] or joint work with Jason Reed [RP10], although I do not believe it has previously been presented in this exact form.

2 Additive Connectives

In linear logic, there are two forms of conjunction. Besides the simultaneous conjunction (called multiplicative), there is an *alternative conjunction* (called additive), written $A \mathbin{\&} B$. We can achieve $A \mathbin{\&} B$ as a goal with the current resources, if we can achieve both A and B with the current resources. This means that if we have a resource $A \mathbin{\&} B$ we can choose to convert this to either A or B.

$$\frac{\Delta;\Gamma \longrightarrow A \quad \Delta;\Gamma \longrightarrow B}{\Delta;\Gamma \longrightarrow A \mathbin{\&} B}\ \&R$$

$$\frac{\Delta;\Gamma, A \longrightarrow C}{\Delta;\Gamma, A \mathbin{\&} B \longrightarrow C}\ \&L_1 \qquad \frac{\Delta;\Gamma, B \longrightarrow C}{\Delta;\Gamma, A \mathbin{\&} B \longrightarrow C}\ \&L_2$$

The resource representation is straightforward, and the proof of the previous lecture will extend naturally.

$$\frac{\Gamma \longrightarrow A \mathrel{@} p \quad \Gamma \longrightarrow B \mathrel{@} p}{\Gamma \longrightarrow A \mathbin{\&} B \mathrel{@} p}\ \&R$$

$$\frac{\Gamma, A \mathbin{\&} B@\alpha, A@\beta \longrightarrow C \mathrel{@} p * \beta}{\Gamma, A \mathbin{\&} B@\alpha \longrightarrow C \mathrel{@} p * \alpha}\ \&L_1^\beta \qquad \frac{\Gamma, A \mathbin{\&} B@\alpha, B@\beta \longrightarrow C \mathrel{@} p * \beta}{\Gamma, A \mathbin{\&} B@\alpha \longrightarrow C \mathrel{@} p * \alpha}\ \&L_2^\beta$$

While the alternative conjunction embodies one form of choice, the disjunction ($A \oplus B$) represents another form of choice. When we have a resource $A \oplus B$ we do not know whether A or B will be provided and we have to account for both possibilities.

$$\frac{\Delta;\Gamma \longrightarrow A}{\Delta;\Gamma \longrightarrow A \oplus B}\ \oplus R_1 \qquad \frac{\Delta;\Gamma \longrightarrow B}{\Delta;\Gamma \longrightarrow A \oplus B}\ \oplus R_2$$

$$\frac{\Delta;\Gamma, A \longrightarrow C \quad \Delta;\Gamma, B \longrightarrow C}{\Delta;\Gamma, A \oplus B \longrightarrow C}\ \oplus L$$

Note that the apparent violation of linearity is not a mistake, as can be seen by verifying the appropriate cases of cut elimination: only one of either A or B can be inferred by a right rule, so only one of the premises of the left rule will come into play in any situation.

$$\frac{\Gamma \longrightarrow A \mathrel{@} p}{\Gamma \longrightarrow A \oplus B \mathrel{@} p}\ \oplus R_1 \qquad \frac{\Gamma \longrightarrow B \mathrel{@} p}{\Gamma \longrightarrow A \oplus B \mathrel{@} p}\ \oplus R_2$$

$$\frac{\Gamma, A \oplus B@\alpha, A@\beta \longrightarrow C \mathrel{@} p * \beta \quad \Gamma, A \oplus B@\alpha, B@\gamma \longrightarrow C \mathrel{@} p * \gamma}{\Gamma, A \oplus B@\alpha \longrightarrow C \mathrel{@} p * \alpha}\ \oplus L^{\beta,\gamma}$$

The identity elements for $\&$ and $\oplus$ are $\top$ and **0**. While $\top$ has only a right rule, **0** has only a left rule. We summarize the rules and their resource formulation.

$$\frac{}{\Delta; \Gamma \longrightarrow \top}\ \top R \qquad \frac{}{\Gamma \longrightarrow \top \mathbin{@} p}\ \top R$$

$$\frac{}{\Delta; \Gamma, \mathbf{0} \longrightarrow C}\ \mathbf{0}L \qquad \frac{}{\Gamma, \mathbf{0} @ \alpha \longrightarrow C \mathbin{@} p * \alpha}\ \mathbf{0}L$$

It is important to remember that **0** must be tethered to the succedent, so we have license to use it.

3 Untethering

While the rules we have presented so far are isomorphic to the sequent calculus rules for linear logic, taken by themselves they have several somewhat questionable aspects. We show three examples and then try to find a unified reformulation that eliminates these.

copy. The copy rule

$$\frac{\Gamma, A@\epsilon, A@\alpha \longrightarrow C \mathbin{@} p * \alpha}{\Gamma, A@\epsilon \longrightarrow C \mathbin{@} p}\ \mathsf{copy}^{\alpha}$$

is sound from the resource perspective, but its necessity can only be explained by reference to the linear sequent calculus. It seems we should be able to directly use any assumption $A@\epsilon$ instead of having to copy it! This suggest a revision where we change all left rules to have a conclusion of the form

$$\Gamma, A@p \longrightarrow C \mathbin{@} p * q$$

where A is the principal formula, $p = \epsilon$ or $p = \alpha$, and p is consumed as part of the inference.

This seems possible, although we have not verified the details at present.

$\& L_i$. We have not discussed a calculus of natural deduction, but clearly the natural rules for the alternative conjunction would be

$$\frac{\Gamma \vdash A \mathbin{@} p \quad \Gamma \vdash B \mathbin{@} p}{\Gamma \vdash A \mathbin{\&} B \mathbin{@} p}\ \& I$$

$$\frac{\Gamma \vdash A \mathbin{\&} B \mathbin{@} p}{\Gamma \vdash A \mathbin{@} p}\ \& E_1 \qquad \frac{\Gamma \vdash A \mathbin{\&} B \mathbin{@} p}{\Gamma \vdash B \mathbin{@} p}\ \& E_2$$

These rules do not quite match our left rules in the resource sequent calculus. The natural rules would be

$$\frac{\Gamma, A \mathbin{\&} B@p, A@p \longrightarrow C \mathbin{@} r}{\Gamma, A \mathbin{\&} B@p \longrightarrow C \mathbin{@} r}\ \&L_1 \qquad \frac{\Gamma, A \mathbin{\&} B@p, B@p \longrightarrow C \mathbin{@} r}{\Gamma, A \mathbin{\&} B@p \longrightarrow C \mathbin{@} r}\ \&L_2$$

where $p = \alpha$ or $p = \epsilon$ (if we take the previous remark about the copy rule into account).

This rule changes one of the fundamental invariants of the system, namely that all resource parameters in the antecedent have at most one occurrence. This affects the translation from the resource calculus to the linear sequent calculus. Recall that if $\Gamma \longrightarrow C \mathbin{@} p$ then $\Gamma|_p \longrightarrow C$, where

$$\begin{array}{lcl} (\cdot)|_\epsilon & = & (\cdot) \\ (\Gamma, \Gamma')|_{p*q} & = & \Gamma|_p, \Gamma'|_q \\ (A@\alpha)|_\alpha & = & A\ \mathit{left} \\ (A@\alpha)|_\epsilon & = & (\cdot) \\ (A@\epsilon)|_\epsilon & = & A\ \mathit{valid} \end{array}$$

This definition remains unchanged, but it is now nondeterministic because if there are multiple assumptions at α, where α occurs in p, then one of them has to be selected (third clause) while all others are ignfored (fourth clause).

We believe the remaining system is still sound and complete, even though we have not checked the details at present. One noteworthy aspect of this system is that the left rule for conjunction is no longer tethered. If we have an assumption $A \mathbin{\&} B@\alpha$ where α does *not* appear in r while proving $C \mathbin{@} r$, then we can still apply the rule, but with no bad consequence. Strengthening will erase all assumptions labeled with α.

With this generalization we can, for example, directly deduce unrestricted assumptions from others. For example, when mapped back to the linear sequent calculus, the instance of $\&L_1$ with $p = \epsilon$ is

$$\frac{\Delta, A \mathbin{\&} B, A; \Gamma \longrightarrow C}{\Delta, A \mathbin{\&} B; \Gamma \longrightarrow C}$$

which is not otherwise derivable (although it is, of course, admissible).

$\multimap L$**.** Again, in linear natural deduction we would expect the rule

$$\frac{\Gamma \vdash A \multimap B \mathbin{@} p \quad \Gamma \vdash A \mathbin{@} q}{\Gamma \vdash B \mathbin{@} p * q}\ \multimap E$$

Compare this with the cumbersome

$$\frac{\Gamma, A \multimap B@\alpha \longrightarrow A \mathbin{@} q \quad \Gamma, A \multimap B@\alpha, B@\beta \longrightarrow C \mathbin{@} p * \beta}{\Gamma, A \multimap B@\alpha \longrightarrow C \mathbin{@} p * q * \alpha} \multimap L^\beta$$

A more direct translation following the connection between natural deduction might be the (untethered)

$$\frac{\Gamma, A \multimap B@p \longrightarrow A \mathbin{@} q \quad \Gamma, A \multimap B@p, B@p * q \longrightarrow C \mathbin{@} r}{\Gamma, A \multimap B@p \longrightarrow C \mathbin{@} r} \multimap L^\beta$$

To write this rule, we must losen our invariants further, allowing not only hypotheses $A@\alpha$ and $A@\epsilon$ but more generally $A@p$ for arbitrary p.

If we allow that, we have to revisit the translation that maps resource hypotheses to linear hypotheses.

$$\begin{array}{lcl} (\cdot)|_\epsilon & = & (\cdot) \\ (\Gamma, \Gamma')|_{p*q} & = & \Gamma|_p, \Gamma'|_q \\ (A@p)|_p & = & A\ \mathit{left} \\ (A@p)|_\epsilon & = & (\cdot) \\ (A@\epsilon)|_\epsilon & = & A\ \mathit{valid} \end{array}$$

There are several sources of nondeterminism; the important part is the total preservation of resources. We expect the following property:

If $\Gamma \longrightarrow C \mathbin{@} p$ then for some $\Psi = \Gamma|_p$ we have $\Psi \longrightarrow C$.

With these observations, we can rewrite the inference rules as shown in Figure 1. We conjecture (but have not checked the details at present) that this resource semantics is sound and complete with respect to linear sequent calculus.

A few observations about this calculus. The rules that require a coordination between the resources in the antecedent and succedent are init (which applies to atomic formulas) as well as the left rules for so-called *positive* propositions ($A \otimes B$, $\mathbf{1}$, $!A$, $A \oplus B$, and $\mathbf{0}$). The left rules for the *negative* propositions ($A \multimap B$, $A \mathbin{\&} B$, $\top$) are untethered and can be applied without reference to the succedent. This means that the negative fragment is particular elegant, as can be seen from its dependent form worked out by Reed [Ree07, Ree09].

A second observation is that the structure of proofs in the untethered sequent calculus is different from the linear sequent calculus. This difference is particularly apparent for the exponentials (empty resource).

$$\frac{}{\Gamma, P@p \longrightarrow P \mathbin{@} p}\ \mathsf{init}$$

$$\frac{\Gamma \longrightarrow A \mathbin{@} p \quad \Gamma \longrightarrow B \mathbin{@} q}{\Gamma \longrightarrow A \otimes B \mathbin{@} p * q}\ {\otimes}R$$

$$\frac{\Gamma, A \otimes B@q, A@\beta, B@\gamma \longrightarrow C \mathbin{@} p * \beta * \gamma}{\Gamma, A \otimes B@q \longrightarrow C \mathbin{@} p * q}\ {\otimes}L^{\beta,\gamma}$$

$$\frac{\Gamma, A@\alpha \longrightarrow B \mathbin{@} p * \alpha}{\Gamma \longrightarrow A \multimap B \mathbin{@} p}\ {\multimap}R^{\alpha}$$

$$\frac{\Gamma, A \multimap B@p \longrightarrow A \mathbin{@} q \quad \Gamma, A \multimap B@p, B@p * q \longrightarrow C \mathbin{@} r}{\Gamma, A \multimap B@p \longrightarrow C \mathbin{@} r}\ {\multimap}L$$

$$\frac{}{\Gamma \longrightarrow \mathbf{1} \mathbin{@} \epsilon}\ \mathbf{1}R \qquad \frac{\Gamma, \mathbf{1}@q \longrightarrow C \mathbin{@} p * q}{\Gamma, \mathbf{1}@q \longrightarrow C \mathbin{@} p}\ \mathbf{1}L$$

$$\frac{\Gamma \longrightarrow A \mathbin{@} \epsilon}{\Gamma \longrightarrow !A \mathbin{@} \epsilon}\ !R \qquad \frac{\Gamma, !A@q, A@\epsilon \Longrightarrow C \mathbin{@} p}{\Gamma, !A@q \Longrightarrow C \mathbin{@} p * q}\ !L$$

$$\frac{\Gamma \longrightarrow A \mathbin{@} p \quad \Gamma \longrightarrow B \mathbin{@} p}{\Gamma \longrightarrow A \mathbin{\&} B \mathbin{@} p}\ \&R$$

$$\frac{\Gamma, A \mathbin{\&} B@p, A@p \longrightarrow C \mathbin{@} r}{\Gamma, A \mathbin{\&} B@p \longrightarrow C \mathbin{@} r}\ \&L_1 \qquad \frac{\Gamma, A \mathbin{\&} B@p, B@p \longrightarrow C \mathbin{@} r}{\Gamma, A \mathbin{\&} B@p \longrightarrow C \mathbin{@} r}\ \&L_2$$

$$\frac{}{\Gamma \longrightarrow \top \mathbin{@} p}\ \top R \qquad \text{no } \top L$$

$$\frac{\Gamma \longrightarrow A \mathbin{@} p}{\Gamma \longrightarrow A \oplus B \mathbin{@} p}\ {\oplus}R_1 \qquad \frac{\Gamma \longrightarrow B \mathbin{@} p}{\Gamma \longrightarrow A \oplus B \mathbin{@} p}\ {\oplus}R_2$$

$$\frac{\Gamma, A \oplus B@q, A@\beta \longrightarrow C \mathbin{@} p * \beta \quad \Gamma, A \oplus B@q, B@\gamma \longrightarrow C \mathbin{@} p * \gamma}{\Gamma, A \oplus B@q \longrightarrow C \mathbin{@} p * q}\ {\oplus}L^{\beta,\gamma}$$

$$\text{no } \mathbf{0}R \qquad \frac{}{\Gamma, \mathbf{0}@q \longrightarrow C \mathbin{@} p * q}\ \mathbf{0}L$$

Figure 1: Untethered resource sequent calculus

We can further modify or extend this calculus in ways which is impossible in linear logic. For example, we can define

$$\frac{\Gamma, A \mathbin{\&} B@p, A@p, B@p \longrightarrow C \mathbin{@} r}{\Gamma, A \mathbin{\&} B@p \longrightarrow C \mathbin{@} r}\ \&L$$

This single rule can replace two rules, $\&L_1$ and $\&L_2$. Unlike those two rules, this one is asynchronous and we can indeed drop $A \mathbin{\&} B@p$ in a so-called *focused* version of this system. This kind of left-invertible rule is impossible for $A \mathbin{\&} B$ in linear logic, because we cannot represent the ties of A and B (when A is used, B becomes unusable, and vice versa).

The fact that we have explicit resources means that we can also define some new connectives that do not exist in linear logic. For example,

$$\frac{\Gamma, A@p \longrightarrow B \mathbin{@} p}{\Gamma \longrightarrow A \rightarrow B \mathbin{@} p}\ {\rightarrow}R$$

$$\frac{\Gamma, A \rightarrow B@p \longrightarrow A \mathbin{@} p \quad \Gamma, A \rightarrow B@p, B@p \longrightarrow C \mathbin{@} r}{\Gamma, A \rightarrow B@p \longrightarrow C \mathbin{@} r}\ {\rightarrow}L$$

Whether the connective above really makes sense, and which other ones might be interesting is a topic for future research. Some hints can be found in the above-cited work by Reed.

Exercises

Exercise 1 *Explain the rules for* $\top$ *and* $\mathbf{0}$.

References

[Ree07] Jason Reed. Hybridizing a logical framework. In *Proceedings of the International Workshop on Hybrid Logic (HyLo'06)*, pages 135–148. Electronic Notes in Theoretical Computer Science, v.174(6), 2007.

[Ree09] Jason Reed. *A Hybrid Logical Framework*. PhD thesis, Carnegie Mellon University, September 2009. Available as Technical Report CMU-CS-09-155.

[RP10] Jason Reed and Frank Pfenning. Focus-preserving embeddings of substructural logics in intuitionistic logic. Draft manuscript, January 2010.