Lecture Notes on
First-Order Modal Logic

15-816: Modal Logic
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1 Introduction to This Lecture

In this lecture, we will introduce first-order modal logic and start considering its relationship to classical first-order logic.

2 Quantified Modal Logic

In this section, we define the syntax and semantics of quantified modal logic [Car46, Kri63]. An excellent source on first-order modal logic, its various variations and pitfalls is the book by Fitting and Mendelsohn [FM99].

We fix a set Σ of function symbols and predicate symbols, with arities associated (number of arguments) and a set of logical variables. Terms are defined as in first-order logic. The syntax of classical first-order modal logic is defined as follows:

Definition 1 (First-order modal formulas) *The set* $\mathrm{Fml}_{FOML}(\Sigma)$ *of* formulas of classical quantified modal logic a.k.a. first-order modal logic *is the smallest set with:*

- *If* $p \in \Sigma$ *is a predicate symbol of arity* n *and* $\theta_1, \dots, \theta_n$ *are terms then* $p(\theta_1, \dots, \theta_n) \in \mathrm{Fml}_{FOML}(\Sigma)$.
- *If* $\phi, \psi \in \mathrm{Fml}_{FOML}(\Sigma)$*, then* $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi) \in \mathrm{Fml}_{FOML}(\Sigma)$.
- *If* $\phi \in \mathrm{Fml}_{FOML}(\Sigma)$ *and* x *is a logical variable, then* $(\forall x\, \phi) \in \mathrm{Fml}_{FOML}(\Sigma)$ *and* $(\exists x\, \phi) \in \mathrm{Fml}_{FOML}(\Sigma)$.

- *If* $\phi \in \mathrm{Fml}_{FOML}(\Sigma)$ *and* $x \in V$*, then* $(\Box\phi), (\Diamond\phi) \in \mathrm{Fml}_{FOML}(\Sigma)$.

There are several variations for the definition of semantics for quantified modal logic. Here is one variant:

Definition 2 (Kripke structure) *A* Kripke structure $K = (W, \rho, M)$ *consists of Kripke frame* (W, ρ) *and a mapping* M *that assigns first-order structures* $M(s)$ *to each world* s *such that, for each* $s, t \in W$ *with* $s\rho t$*, the structure* $M(s)$ *is a substructure of* $M(t)$*, i.e.:*

- *the universe of* $M(s)$ *is a subset of the universe of* $M(t)$ *(monotonicity), and*
- *the structures* $M(s)$ *and* $M(t)$ *agree on the interpretation of all* function symbols *on the (smaller) universe of* $M(s)$.

Another common case in the semantics is that of *constant domain*, where all worlds in a Kripke structure are required to share the same universe.

Definition 3 (Interpretation of quantified modal formulas) *Given a Kripke structure* $K = (W, \rho, M)$*, the interpretation* $\models$ *of modal formulas in a world* s *is defined as*

1. $K, s \models p(\theta_1, \ldots, \theta_n)$ *iff* $M(s) \models p(\theta_1, \ldots, \theta_n)$.
2. $K, s \models \phi \land \psi$ *iff* $K, s \models \phi$ *and* $K, s \models \psi$.
3. $K, s \models \phi \lor \psi$ *iff* $K, s \models \phi$ *or* $K, s \models \psi$.
4. $K, s \models \neg\phi$ *iff it is not the case that* $K, s \models \phi$.
5. $K, s \models \forall x\, \phi(x)$ *iff* $K, s \models \phi(d)$ *for all* d *in the universe of* s
6. $K, s \models \exists x\, \phi(x)$ *iff* $K, s \models \phi(d)$ *for some* d *in the universe of* s
7. $K, s \models \Box\phi$ *iff* $K, t \models \phi$ *for all worlds* t *with* $s\rho t$.
8. $K, s \models \Diamond\phi$ *iff* $K, t \models \phi$ *for some world* t *with* $s\rho t$.

When K *is clear from the context, we sometimes abbreviate* $K, s \models \phi$ *by* $s \models \phi$.

In constant domain semantics, quantifiers refer to the same set of objects in all worlds. In varying domain semantics, quantifiers may possibly refer to a different set of objects, depending on the world.

Exercises

Exercise 1 *Recall Definition 3 of interpretation of quantified modal formulas. The definition is imprecise at some points. What is the problem, why is it a problem, and what can be done to fix it?*

References

[Car46] Rudolf Carnap. Modalities and quantification. *J. Symb. Log.*, 11(2):33–64, 1946.

[FM99] Melvin Fitting and Richard L. Mendelsohn. *First-Order Modal Logic*. Kluwer, Norwell, MA, USA, 1999.

[Kri63] Saul A. Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.