Recitation 8: Bidirectional Typechecking and Sequent Calculus

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1 Bidirectional Typechecking

The proof term assignment for the logic of verifications and uses can be construed as an algorithmic specification for a typechecker; this is because the rules of this logic are all *syntax-directed*.

To expose the algorithmic character of the judgments, we write in blue the *inputs* of a judgment and in red the *outputs* of a judgment.

$$\frac{}{() : \top \uparrow}\ \top\mathsf{I} \qquad \frac{R : \bot \downarrow}{\mathsf{abort}(R) : C \uparrow}\ \bot\mathsf{E} \qquad \frac{M : A \uparrow \quad N : B \uparrow}{(M, N) : A \wedge B \uparrow}\ \wedge\mathsf{I}$$

$$\frac{R : A \wedge B \downarrow}{\mathsf{fst}(R) : A \downarrow}\ \wedge\mathsf{E}_1 \qquad \frac{R : A \wedge B \downarrow}{\mathsf{snd}(R) : B \downarrow}\ \wedge\mathsf{E}_2 \qquad \frac{M : A \uparrow}{\mathsf{inl}(M) : A \vee B \uparrow}\ \vee\mathsf{I}_1$$

$$\frac{M : B \uparrow}{\mathsf{inr}(M) : A \vee B \uparrow}\ \vee\mathsf{I}_2 \qquad \frac{R : A \vee B \downarrow \quad \begin{array}{c}\overline{u : A \downarrow}^{\,u} \\ \vdots \\ N_1 : C \uparrow\end{array} \quad \begin{array}{c}\overline{v : B \downarrow}^{\,v} \\ \vdots \\ N_2 : C \uparrow\end{array}}{\mathsf{case}\ R\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow N_1 \mid \mathsf{inr}(v) \Rightarrow N_2 : C \uparrow}\ \vee\mathsf{E}^{u,v}$$

$$\frac{\begin{array}{c}\overline{u : A \downarrow}^{\,u} \\ \vdots \\ N : B \uparrow\end{array}}{\mathsf{fn}\ u \Rightarrow N : A \supset B \uparrow}\ \supset\mathsf{I}^u \qquad \frac{R : A \supset B \downarrow \quad N : A \uparrow}{R(N) : B \downarrow}\ \supset\mathsf{E}$$

$$\frac{R : A \downarrow}{R : A \uparrow}\ \downarrow\uparrow \qquad \frac{M : A \uparrow}{(M : A) : A \downarrow}\ \mathsf{ann}$$

If the last rule ann is omitted, then the terms of the verifications and uses calculus will have no redexes; if it is added, then redexes can be formed, and the calculus is merely a more verbose version of the ordinary λ-calculus. This week, we are studying a version of verifications and uses with the ann rule, because it is necessary in order to give a natural proof term assignment to Dyckhoff's contraction-free sequent calculus.

the algorithm We will use $\Delta \vdash \mathcal{J}$ to abbreviate $\begin{matrix}\Delta \\ \vdots \\ \mathcal{J}\end{matrix}$ in what follows, where Δ is a sequence of assumptions $\overline{x : A\downarrow}$. The typechecking algorithm for bidirectional typechecking is the computational content of the following (constructive) metatheorem:

1. For all Δ, N, A, either $\Delta \vdash N : A\uparrow$ or not.
2. For all Δ, R, either there exists some A such that $\Delta \vdash R : A\downarrow$ or there does not.

Exercise. *Try and remember how this proof works. You will have to extract its algorithmic content in order to complete the next homework assignment.*

Remark. *This metatheorem is only interesting if the ambient mathematics is constructive; otherwise, it is trivial and its proof provides no useful algorithmic content.*

2 Proof terms for sequent calculus

We can give a proof term assignment to the sequent calculus based on a single form of judgment $\Gamma \Longrightarrow N : A$; in this form of judgment, Γ is now a list of formal type assignments $R : A$, where R is an arbitrary neutral term rather than only a variable. The dynamics are as follows:

1. The list of type assignments $\Gamma \equiv \overline{R : A}$ are inputs.
2. The goal type A is an input.
3. The proof term / witness N is an output.

These dynamics correspond to *proof refinement with extraction* in sequent-calculus-based proof systems like **Nuprl** and **RedPRL**.

$$\frac{}{\Gamma, R : A \Longrightarrow R : A}\ \mathsf{init} \qquad \frac{}{\Gamma \Longrightarrow () : \top}\ \top\mathsf{R} \qquad \frac{}{\Gamma, R : \bot \Longrightarrow \mathsf{abort}(R) : C}\ \bot\mathsf{L}$$

$$\frac{\Gamma \Longrightarrow N_1 : A}{\Gamma \Longrightarrow (N_1, N_2) : A \wedge B}\ \wedge\mathsf{R} \qquad \frac{\Gamma, R : A \wedge B, \mathsf{fst}(R) : A \Longrightarrow N : C}{\Gamma, R : A \wedge B \Longrightarrow N : C}\ \wedge\mathsf{L}_1$$

$$\frac{\Gamma, R : A \wedge B, \mathsf{snd}(R) : B \Longrightarrow N : C}{\Gamma, R : A \wedge B \Longrightarrow N : C}\ \wedge\mathsf{L}_2 \qquad \frac{\Gamma \Longrightarrow N : A}{\Gamma \Longrightarrow \mathsf{inl}(N) : A \vee B}\ \vee\mathsf{R}_1$$

$$\frac{\Gamma \Longrightarrow N : B}{\Gamma \Longrightarrow \mathsf{inr}(N) : A \vee B}\ \vee\mathsf{R}_2$$

$$\frac{\Gamma, R : A \vee B, u : A \Longrightarrow N_1 : C \qquad \Gamma, R : A \vee B, v : A \Longrightarrow N_2 : C}{\Gamma, R : A \vee B \Longrightarrow \mathsf{case}\ R\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow N_1 \mid \mathsf{inr}(v) \Rightarrow N_2 : C}\ \vee\mathsf{L}^{u,v}$$

$$\frac{\Gamma, u : A \Longrightarrow N : B}{\Gamma \Longrightarrow \mathsf{fn}\ u \Rightarrow N : A \supset B} \supset\mathsf{R}^u$$

$$\frac{\Gamma, R : A \supset B \Longrightarrow M : A \quad \Gamma, R : A \supset B, R(M) : B \Longrightarrow N : C}{\Gamma, R : A \supset B \Longrightarrow N : C} \supset\mathsf{L}$$

Exercise. *In sequent calculus, we try to prove the admissibility of the following* cut *rule:*

$$\frac{\Gamma \Longrightarrow A \quad \Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow B} \mathsf{cut}$$

Try to invent a proof term assignment for this rule.

Solution.

$$\frac{\Gamma \Longrightarrow M : A \quad \Gamma, (M : A) : A \Longrightarrow N : B}{\Gamma \Longrightarrow N : B} \mathsf{cut}_A$$