Recitation 3: Harmony

Course Staff

Proof-theoretic harmony is a necessary, but not sufficient, condition for the well-behavedness of a logic; harmony ensures that the connectives are *locally* well-behaved, and is closely related to the critical cases of cut and identity elimination which we may discuss later on. Therefore, when designing or extending a logic, checking harmony is a first step.

From the verificationist standpoint, a connective is *harmonious* if its elimination rules are neither too strong nor too weak in relation to its introduction rules. The first condition is called *local soundness* and the second condition is called *local completeness*. The content of the soundness condition is a method to reduce or simplify proofs, and the content of completeness is a method to expand any arbitrary proof into a canonical proof (i.e. one that ends in an introduction rule).

1 Conjunction

Local soundness for conjunction is witnessed by the following two reduction rules:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\,\wedge\mathsf{I}}{A\ \mathit{true}}\,\wedge\mathsf{E}_1 \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\,\wedge\mathsf{I}}{B\ \mathit{true}}\,\wedge\mathsf{E}_2 \quad \longrightarrow_{\mathsf{R}} \quad \begin{array}{c}\mathcal{E}\\ B\ \mathit{true}\end{array}$$

Local completeness is witnessed by the following expansion rule:

$$\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array} \quad \longrightarrow_{\mathsf{E}} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array}}{A\ \mathit{true}}\,\wedge\mathsf{E}_1 \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\ \mathit{true}\end{array}}{B\ \mathit{true}}\,\wedge\mathsf{E}_2}{A \wedge B\ \mathit{true}}\,\wedge\mathsf{I}$$

When regarded as generating relations on *programs* rather than proofs, the reduction and expansion rules can be recast into another familiar format:

$$\begin{aligned}
\mathsf{fst}(\langle M, N\rangle) &\longrightarrow_{\mathsf{R}} M\\
\mathsf{snd}(\langle M, N\rangle) &\longrightarrow_{\mathsf{R}} N\\
M &\longrightarrow_{\mathsf{E}} \langle \mathsf{fst}(M), \mathsf{snd}(M)\rangle
\end{aligned}$$

2 Disjunction

Instructions: present local soundness for proofs, and ask the students to come up with the version for programs. Next, elicit from the students both local completeness for programs and for proofs.

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_1 \quad \boxed{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\lor\mathsf{E}^{u,v} \quad \longrightarrow_{\mathsf{R}} \boxed{\begin{array}{c}\boxed{\mathcal{D}}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array}}$$

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ B\ \mathit{true}\end{array}}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_2 \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{E}\\ C\ \mathit{true}\end{array} \quad \boxed{\begin{array}{c}\overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}}{C\ \mathit{true}}\lor\mathsf{E}^{u,v} \quad \longrightarrow_{\mathsf{R}} \boxed{\begin{array}{c}\boxed{\mathcal{D}}\\ \overline{B\ \mathit{true}}^{\,v}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}$$

$$\mathsf{case}\ \mathsf{inl}(\boxed{M})\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow \boxed{L} \mid \mathsf{inr}(v) \Rightarrow R \longrightarrow_{\mathsf{R}} \boxed{[\boxed{M} \,/\, u]L}$$

$$\mathsf{case}\ \mathsf{inr}(\boxed{M})\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow L \mid \mathsf{inr}(v) \Rightarrow \boxed{R} \longrightarrow_{\mathsf{R}} \boxed{[\boxed{M} \,/\, v]R}$$

$$\boxed{\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array}} \longrightarrow_{\mathsf{E}} \dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array}} \quad \dfrac{A\ \mathit{true}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_1 \quad \dfrac{B\ \mathit{true}}{A \lor B\ \mathit{true}}\lor\mathsf{I}_2}{A \lor B\ \mathit{true}}\lor\mathsf{E}^{u,v}$$

$$\boxed{M} \longrightarrow_{\mathsf{E}} \mathsf{case}\ \boxed{M}\ \mathsf{of}\ \mathsf{inl}(u) \Rightarrow \mathsf{inl}(u) \mid \mathsf{inr}(v) \Rightarrow \mathsf{inr}(v)$$

3 Implication

Elicit both local soundness and local completeness from students in both proof and program notation.

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^{u} \quad \boxed{\begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array}}}{B}\supset\mathsf{E} \quad \longrightarrow_{\mathsf{R}} \boxed{\begin{array}{c}\boxed{\mathcal{E}}\\ \overline{A\ \mathit{true}}^{\,u}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}$$

$$(\mathsf{fn}\ u \Rightarrow \boxed{M})(\boxed{N}) \longrightarrow_{\mathsf{R}} \boxed{[\boxed{N} \,/\, u]M}$$

$$\boxed{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array}} \longrightarrow_{\mathsf{E}} \dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array}} \quad \overline{A\ \mathit{true}}^{\,u}}{B\ \mathit{true}}\supset\mathsf{E}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^{u}$$

$$\boxed{M} \longrightarrow_{\mathsf{E}} \mathsf{fn}\ u \Rightarrow \boxed{M}(u)$$

4 Experiment: Alternative Implication

What if we replaced the $\supset\mathsf{E}$ rule with the following elimination rule:

$$\dfrac{A \supset B\ \mathit{true} \quad A\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}} \supset\!\mathsf{E}^u$$

The program/proof term assignment is as follows:

$$\dfrac{L : A \supset B \quad M : A \quad \begin{array}{c}\overline{u : B}^{\,u}\\ \vdots \\ N : C\end{array}}{\mathsf{let}\ u = L(M)\ \mathsf{in}\ N : C} \supset\!\mathsf{E}^u$$

Can we show local soundness and completeness for this version of the implication connective?

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\overline{A\ \mathit{true}}^{\,v}\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}}{A \supset B\ \mathit{true}}\supset\!\mathsf{I}^v \quad \boxed{\begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array}} \quad \boxed{\begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}}{C\ \mathit{true}}\supset\!\mathsf{E}^u \longrightarrow_{\mathsf{R}} \boxed{\begin{array}{c}\boxed{\begin{array}{c}\dfrac{\boxed{\begin{array}{c}\mathcal{E}\\ A\ \mathit{true}\end{array}}}{A\ \mathit{true}}\,v\\ \mathcal{D}\\ B\ \mathit{true}\end{array}}\\ \overline{B\ \mathit{true}}\,u\\ \mathcal{F}\\ C\ \mathit{true}\end{array}}$$

$$\mathsf{let}\ u = \boxed{\mathsf{fn}\ v \Rightarrow L}(\boxed{M})\ \mathsf{in}\ \boxed{N} \longrightarrow_{\mathsf{R}} \boxed{[\boxed{[\boxed{M}\ /\ v]L}\ /\ u]N}$$

$$\boxed{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array}} \longrightarrow_{\mathsf{E}} \dfrac{\dfrac{\boxed{\begin{array}{c}\mathcal{D}\\ A \supset B\ \mathit{true}\end{array}} \quad \overline{A\ \mathit{true}}^{\,u} \quad \overline{B\ \mathit{true}}^{\,v}}{B\ \mathit{true}}\supset\!\mathsf{E}^v}{A \supset B\ \mathit{true}}\supset\!\mathsf{I}^u$$

$$\boxed{M} \longrightarrow_{\mathsf{E}} \mathsf{fn}\ u \Rightarrow \mathsf{let}\ v = \boxed{M}(u)\ \mathsf{in}\ v$$