Assignment 4 Solution: Classical Logic

15-317: Constructive Logic

Out: Thursday, September 25, 2008
Due: Thursday, October 2, 2008, before class

Much of this solution is copied from Anton Bachin's work—thanks Anton!

This file contains the classical natural deduction proofs; see the Tutch solutions for the double-negation translations and their proofs.

1. Natural deduction tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vee \neg P \ \mathsf{false}}\,u \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \ \mathsf{true}}\,w}{P \vee \neg P \ \mathsf{true}}\,\vee I_L}{\#}\,fE}{P \ \mathsf{false}}\,fI^w}{\neg P \ \mathsf{true}}\,\neg I}{P \vee \neg P \ \mathsf{true}}\,\vee I_R}{\#}\,fE}{P \vee \neg P \ \mathsf{true}}\,DNE^u$$

2. Natural deduction tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \ \mathsf{false}}\,w \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P \ \mathsf{false}}\,w \quad \dfrac{}{P \ \mathsf{true}}\,v}{\#}\,fE}{Q \ \mathsf{true}}\,\#E}{P \supset Q \ \mathsf{true}}\,\supset I^v}{(P \supset Q) \supset P \ \mathsf{true}}\,u}{P \ \mathsf{true}}\,\supset E}{\#}\,fE}{P \ \mathsf{true}}\,DNE^w}{((P \supset Q) \supset P) \supset P \ \mathsf{true}}\,\supset I^u$$

3. Natural deduction tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(P \wedge Q)\ \mathsf{true}}\,u}{P \wedge Q\ \mathsf{false}}\,\neg E \qquad \dfrac{\dfrac{\dfrac{}{\neg P \vee \neg Q\ \mathsf{false}}\,w \quad \dfrac{\dfrac{\dfrac{}{P\ \mathsf{false}}\,v}{\neg P\ \mathsf{true}}\,\neg I}{\neg P \vee \neg Q\ \mathsf{true}}\,\vee I_L}{\#}\,fE}{P\ \mathsf{true}}\,DNE^v \qquad \dfrac{\dfrac{\dfrac{}{\neg P \vee \neg Q\ \mathsf{false}}\,w \quad \dfrac{\dfrac{\dfrac{}{Q\ \mathsf{false}}\,x}{\neg Q\ \mathsf{true}}\,\neg I}{\neg P \vee \neg Q\ \mathsf{true}}\,\vee I_R}{\#}\,fE}{Q\ \mathsf{true}}\,DNE^x}{P \wedge Q\ \mathsf{true}}\,\wedge I}{\#}\,fE}{\neg(P \wedge Q) \supset (\neg P \vee \neg Q)\ \mathsf{true}}\,\supset I^u
$$

Note that we have used the derived rule

$$
\dfrac{\neg A\ \mathsf{true}}{A\ \mathsf{false}}\,\neg E
$$

which abbreviates

$$
\dfrac{\neg A\ \mathsf{true} \quad \dfrac{}{A\ \mathsf{false}}\,k}{A\ \mathsf{false}}\,\neg E^k
$$

4. Natural deduction tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P \vee Q\ \mathsf{false}}\,w \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P \supset Q\ \mathsf{true}}\,u \quad \dfrac{}{P\ \mathsf{true}}\,v}{Q\ \mathsf{true}}\,\supset E}{\neg P \vee Q\ \mathsf{true}}\,\vee I_R}{\#}\,fE}{P\ \mathsf{false}}\,fI^v}{\neg P\ \mathsf{true}}\,\neg I}{\neg P \vee Q\ \mathsf{true}}\,\vee I_L \quad \dfrac{}{\neg P \vee Q\ \mathsf{false}}\,w}{\#}\,fE}{\neg P \vee Q\ \mathsf{true}}\,DNE^w}{(P \supset Q) \supset (\neg P \vee Q)\ \mathsf{true}}\,\supset I^u
$$

5. The natural deduction tree is split into parts due to space constraints. Let $\mathcal{D}$ be this derivation from assumptions w and z:

$$
\dfrac{\dfrac{\dfrac{}{(P \supset Q)b \vee (P \supset R)\ \mathsf{false}}\,w \quad \dfrac{\dfrac{\dfrac{}{R\ \mathsf{true}}\,z}{P \supset R\ \mathsf{true}}\,\supset I}{(P \supset Q) \vee (P \supset R)\ \mathsf{true}}\,\vee I_R}{\#}\,fE}{Q\ \mathsf{true}}\,\#E
$$

Then, the natural deduction tree for the entire proof is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q) \vee (P \supset R)\ \mathsf{false}}\,w \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \supset (Q \vee R)\ \mathsf{true}}\,u \quad \dfrac{}{P\ \mathsf{true}}\,x}{Q \vee R\ \mathsf{true}}\,\supset E \quad \dfrac{}{Q\ \mathsf{true}}\,y \quad \begin{matrix}\mathcal{D}\\ Q\end{matrix}}{Q\ \mathsf{true}}\,\vee E^{y,z}}{P \supset Q\ \mathsf{true}}\,\supset I^{v}}{(P \supset Q) \vee (P \supset R)\ \mathsf{true}}\,\vee I_L}{\#}\,fE}{(P \supset Q) \vee (P \supset R)\ \mathsf{true}}\,DNE^{w}}{(P \supset (Q \vee R)) \supset ((P \supset Q) \vee (P \supset R))\ \mathsf{true}}\,\supset I^{u}
$$

6. Natural deduction tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q) \vee (Q \supset P)\ \mathsf{false}}\,u \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \supset Q) \vee (Q \supset P)\ \mathsf{false}}\,u \quad \dfrac{\dfrac{\dfrac{}{Q\ \mathsf{true}}\,v}{P \supset Q\ \mathsf{true}}\,\supset I^{x}}{(P \supset Q) \vee (Q \supset P)\ \mathsf{true}}\,\vee I_L}{\#}\,fE}{P\ \mathsf{true}}\,\#E}{Q \supset P\ \mathsf{true}}\,\supset I^{v}}{(P \supset Q) \vee (Q \supset P)\ \mathsf{true}}\,\vee I_R}{\#}\,fE}{(P \supset Q) \vee (Q \supset P)\ \mathsf{true}}\,DNE^{u}
$$