Stable and Sequential Functions on Scott domains, dI-domains and FM-domains

Stephen Brookes
Shai Geva

School of Computer Science
Carnegie Mellon University
Background

- Plotkin: the full abstraction problem for a sequential functional programming language PCF: start of search for semantic characterization of sequential functions.

- Kahn, Plotkin: *sequential functions* on concrete data structures (and concrete domains), using cell structure. Not closed under sequential function space.

- Berry: *stable functions* on dI-domains, and *stable ordering*. A cartesian closed category, but stability does not imply sequentiality.


- Berry, Curien: *sequential algorithms* on concrete data structures. A cartesian closed category, but not extensional, does not solve full abstraction for PCF. Sequentiality based on cell structure.

- Bucciarelli, Ehrhard: *sequential algorithms* on sequential structures. A cartesian closed category, but does not solve PCF problem. Sequentiality based on extra coherence structure.

- None of these definitions permits a characterization of sequentiality in an arbitrary Scott domain.
Our Contribution

• A new definition of sequential functions for Scott domains, characterized by a generalized form of topology. Sequentiality defined intrinsically.

• Considerably expands the class of domains for which sequential functions may be defined.

• Our sequential functions coincide with Kahn-Plotkin sequential functions when restricted to distributive concrete domains.

• The sequential functions between two dI-domains, ordered stably, form a dI-domain.

• The category of dI-domains and sequential functions is not cartesian closed: application is not sequential. We attribute this to certain operational assumptions underlying our notion of sequentiality.

• Scott domains satisfying a “finite meet” property are closed under the pointwise-ordered stable function space, so that we obtain a new stable model based on the pointwise order.

• Towards a class of domains closed under pointwise-ordered sequential function space...and perhaps a solution to the full abstraction problem for PCF?
Generalized Topologies

A generalized topological framework $\Omega$ assigns to each domain $D$ a family $\Omega D$ of subsets of $D$, called $\Omega$-open sets, together with an ordering relation $\leq_\Omega$ on $\Omega D$.

- We define the $\Omega$-continuous functions from $D$ to $E$ to be the functions $f$ such that the inverse image $f^{-1}(q)$ of every $q \in \Omega E$ is in $\Omega D$.
- We will order these functions by $f \leq_\Omega g$ iff for every $q \in \Omega E$, $f^{-1}(q) \leq_\Omega g^{-1}(q)$.
- Different orders on $\Omega$-opens will naturally induce different orders on the $\Omega$-continuous functions.
- We obtain a category of domains and $\Omega$-continuous functions: the identity function is always $\Omega$-continuous, and composition preserves $\Omega$-continuity.
- We are mainly interested in showing that a class of domains is closed under $\Omega$-continuous function space. A necessary condition (not always sufficient) is that $(\Omega D, \leq_\Omega)$ belong to the class of domains whenever $D$ does.
Remarks

• $\Omega D$ is a topology if
  
  – $\emptyset$ and $D$ are $\Omega$-open;
  
  – $\Omega$-open sets are closed under arbitrary unions and finite intersections;
  
  – The order on $\Omega D$ is set inclusion.

• Equivalently, if $\Omega D$ is a sub-frame of the powerset lattice of $D$, ordered by inclusion.
The Scott Topology

As is well known...

- A set $p \subseteq D$ is Scott open iff it is upwards closed and for every directed set $X$, if $\forall X \in p$ then $x \in p$ for some $x \in X$.
- We write $\text{Sc}D$ for the set of Scott opens of $D$.
- Scott opens, ordered by inclusion, determine the Scott topology.
- For every $x \in D_{\text{fin}}$, $\text{up}(x)$ is Scott open.
- $p$ is Scott open iff $p = \cup \{\text{up}(x) \mid x \in p \cap D_{\text{fin}}\}$.
- A function $f : D \rightarrow E$ is Scott continuous, or just continuous, iff the inverse image of every Scott open is Scott open.
- Equivalently, a function $f : D \rightarrow E$ is continuous iff it is monotone and preserves directed lubs.
- Set inclusion on Scott opens induces an order on continuous functions: $f \leq g$ iff
  \[ \forall q \in \text{Sc}E. f^{-1}(q) \subseteq g^{-1}(q). \]
  This is the pointwise order: $f \leq g$ iff $\forall x \in D. f(x) \leq g(x)$.
Stable Opens and Stable Functions

• A set \( p \subseteq D \) is stable iff it is closed under consistent meets, i.e., \( x_1, x_2 \in p \) and \( x_1 \uparrow x_2 \) imply \( x_1 \land x_2 \in p \).

• A set \( p \) is stable open iff it is Scott open and stable.

• We write St\( D \) for the set of stable opens of \( D \).

• For any \( x \in D_{\text{fin}} \), up\( (x) \) is stable open.

• A function \( f : D \rightarrow E \) is stable continuous, or stable, iff the inverse image of every stable open is stable open.

• For a function \( f : D \rightarrow E \), the following are equivalent:
  
  (1) \( f \) is stable.
  
  (2) \( f \) is continuous and preserves consistent meets:
      if \( x_1 \uparrow x_2 \) then \( f(x_1 \land x_2) = f(x_1) \land f(x_2) \).
  
  (3) \( f \) is continuous and whenever \( e \leq f(d) \), the set
      \( \{ d' \in D \mid d' \leq d \ \& \ e \leq f(d') \} \) is down-directed.

• Definition (3) specializes in dI-domains to the usual “minimum point” definition of stable functions: \( f \) is stable iff it is continuous and for every \( e \leq f(d) \) the set \( \{ d' \leq d \mid e \leq f(d') \} \) has a least element.

• Our treatment extends Zhang’s characterization of “stable neighborhoods”.


Scott is not always stable

• Every stable open is also Scott open, by definition.
• The converse fails. For example, the Scott open set
  \[ \text{up}(\{(\top, \bot), (\bot, \top)\}) \subseteq 2 \times 2, \]
  is not stable, because it does not contain
  \[ (\bot, \bot) = (\top, \bot) \land (\bot, \top), \]
  and this is a consistent meet.
• Every stable function is also Scott continuous.
• The converse fails. For example, the parallel-or function is continuous but not stable. The inverse image
  \[ \text{por}^{-1}(\{\text{tt}\}) = \{(\text{tt}, \bot), (\bot, \text{tt})\} \]
  is not stable open.
Lobes of a Stable Set

- A stable set $p$ can be partitioned by identifying all pairs of points of $p$ that have a lower bound in $p$.
- We call the equivalence classes the *lobes* of $p$.
- A lobe is downwards-directed.
- In a dI-domain every lobe has a least element.
- In a Scott domain lobes may fail to contain their glb.
Covering, covers and indices

• The covering relation between elements of $D$ is: $x \prec y$ iff $x < y$ and there is no point between $x$ and $y$.

• A cover of $x \in D$ is a stable set $r$ such that $x < y$ for every $y \in r$ and $\Delta(x, r) = \emptyset$, where

$$\Delta(x, r) = \{ z \mid x < z & \exists r' \in \text{lobes}(r) . \forall y \in r' . z < y \} .$$

We write $I(x)$ for the set of covers of $x$.

• Equivalently, a stable set $r$ is a cover of $x$ iff for every lobe $r'$ of $r$, either $r'$ has a least element $y$ and $x \prec y$, or $r'$ has no least element and $x = \bigwedge r'$.

• For $x \in D$ and $s \subseteq D$, an index of $s$ at $x$ is a cover $r$ of $x$ such that $s \cap \text{up}(x) \subseteq r$.

• Let $I(x, s)$ be the set of indices of $s$ at $x$:

$$I(x, s) = \{ r \in I(x) \mid s \cap \text{up}(x) \subseteq r \} .$$
Intuition

- A stable set $s$ represents a choice between its lobes.
- If the current state of information is $x$, a cover of $x$ represents an atomic increase in information content, with atomicity captured by the condition $\Delta(x, r) = \emptyset$.
- A cover $r$ of $x$ provides a way of locally decomposing the domain at $x$ into a flat domain, with $x$ as the least element and the lobes of $r$ as the proper elements.
- Covers may be used to reason about the progress of an incremental computation, generalizing the notion of cell in a concrete data structure.
- The existence of an index $r \in \llbracket x, s \rrbracket$ indicates that the choice represented by $s$ may be decomposed, with the index $r$ serving as a first step from $x$ towards $s$.

Some Obvious Properties

- $\Delta(x, \emptyset) = \emptyset$.
- $\Delta(x, r) = \bigcup \{ \Delta(x, r') \mid r' \in \text{lobes}(r) \}$.
- $\emptyset \in \llbracket x, \emptyset \rrbracket$.
- $\llbracket x, s \rrbracket = \llbracket x, s \cap \text{up}(x) \rrbracket$. 

11
Stable is not always sequential

- In these domains the shaded points form a stable open set with no index at $\perp$, since the shaded points are not contained in any cover of $\perp$.

- Another example of a stable open with no index at $\perp$:

  \[
  \text{up}\{(tt, ff, \perp), (\perp, tt, ff), (ff, \perp, tt)\} \subseteq \text{Bool} \times \text{Bool} \times \text{Bool}.
  \]

- Absence of an index implies non-sequentiality...
Sequential Opens

• A set $p \subseteq D$ is sequential at $x \in D$ iff $x \in p$, or $x \notin p$ and for every finite $s \subseteq p$, $I(x, s) \neq \emptyset$.

• A set $p$ is sequential iff it is sequential at every $x \in D_{\text{fin}}$.

• A sequential open is a stable open that is sequential.

• We write $SqD$ for the set of sequential opens of $D$.

• For any $x \in D_{\text{fin}}$, $\text{up}(x)$ is sequential open.

• If $x < y$ then $I(x, \text{up}(y)) \neq \emptyset$.

Sequential Functions

• A function $f : D \to E$ is sequential iff the inverse image of every sequential open is sequential open.

Properties

• Every sequential function is Scott-continuous.

• Every sequential function is stable.
Examples

• The doubly-strict-or function \( \text{sor} : \text{Bool}^2 \rightarrow \text{Bool} \) is sequential (and stable).
  
  – The inverse image of the sequential open set \( \{tt\} \) is the sequential open set \( p = \{(tt, tt), (tt, ff), (ff, tt)\} \).
  
  – There are two indices of \( p \) at \( (\bot, \bot) \): \( \text{up}(\{(tt, \bot), (ff, \bot)\}) \) and \( \text{up}(\{(\bot, tt), (\bot, ff)\}) \).
  
  – These two indices at \( (\bot, \bot) \) correspond to the fact that this function is strict in both arguments.

• The left-strict-or function \( \text{lor} \) is also sequential. There is a single index \( \text{up}(\{(tt, \bot), (ff, \bot)\}) \) for \( \text{lor}^{-1}(\{tt\}) \) at \( (\bot, \bot) \).

• The parallel-or function \( \text{por} : \text{Bool}^2 \rightarrow \text{Bool} \) is not sequential, since the inverse image of \( \{tt\} \) is not sequential open (and not even stable).
Stable is not always sequential

• Let $gf : \text{Bool}^3 \to \text{Bool}$ be the least continuous function such that

$$
gf(tt, ff, \bot) = tt$$
$$
gf(\bot, tt, ff) = tt$$
$$
gf(ff, \bot, tt) = tt$$
$$
gf(ff, ff, ff) = ff.
$$

This function is stable but not sequential. The stable open set $gf^{-1}(\{tt\}) = \text{up}(\{(tt, ff, \bot), (ff, \bot, tt), (\bot, tt, ff)\})$ is not sequential open, since it has no index at $(\bot, \bot, \bot)$.

• Let $gf_1, gf_2, gf_3 : \text{Bool}^3 \to \text{Bool}$ map $(ff, ff, ff)$ to $ff$, and satisfy

$$
gf_1(tt, ff, \bot) = tt$$
$$
gf_2(\bot, tt, ff) = tt$$
$$
gf_3(ff, \bot, tt) = tt.
$$

Let their pairwise lubs be $gf_{1,2} = gf_1 \lor gf_2$, $gf_{1,3} = gf_1 \lor gf_3$, and $gf_{2,3} = gf_2 \lor gf_3$. All of these functions are sequential.

• Since $gf = gf_1 \lor gf_2 \lor gf_3$, this shows that a pairwise consistent set of sequential functions need not have a sequential lub. This works with either stable or pointwise order, since the orders coincide in this case. As a corollary, concrete domains are not closed under sequential function space.
Products

• The categories of Scott domains and (respectively) continuous, stable and sequential functions are cartesian.

• The projection functions $\pi_i : D_1 \times D_2 \rightarrow D_i$, for $i = 1, 2$, are sequential.

• For Scott domains $D_1$ and $D_2$,

$$\text{Sc}(D_1 \times D_2) = \{ p_1 \times p_2 \mid p_1 \in \text{Sc}D_1 \text{ and } p_2 \in \text{Sc}D_2 \}$$

$$\text{St}(D_1 \times D_2) \supseteq \{ p_1 \times p_2 \mid p_1 \in \text{St}D_1 \text{ and } p_2 \in \text{St}D_2 \}$$

$$\text{Sq}(D_1 \times D_2) \supseteq \{ p_1 \times p_2 \mid p_1 \in \text{Sq}D_1 \text{ and } p_2 \in \text{Sq}D_2 \}$$

• Stable or sequential opens of $D_1 \times D_2$ may not be formed by a product of stable or sequential opens of $D_1$ and $D_2$.

• For example, let $p = \text{up} \{(\text{tt}, \bot), (\text{tt}, \text{tt})\} \cup \{((\bot, \text{tt}), (\bot, \text{tt})\}$. While $p$ is stable and sequential, $\pi_1(p) = \text{up} \{(\text{tt}, \bot), (\bot, \text{tt})\}$ is neither stable nor sequential.
**Relationship to Kahn-Plotkin**

In a distributive concrete domain $D$,

1. Every non-empty cover $r$ of $x$ corresponds to a unique cell $c$ accessible from $x$ and filled in all elements of $r$.

2. For every Scott open $p$ and $x \notin p$, every finite subset $s$ of $p$ has an index at $x$ iff $p$ itself has an index at $x$.

3. For every sequential open $p$ the set $C$ of cells that are filled in all elements of $p$ is finite. If $p \neq \emptyset$ and $p \neq \uparrow \bot$, $C$ is non-empty.

   For every finite set of cells $C$, the set of states that fill all cells in $C$ is sequential open.

4. A Scott open $p$ is sequential at every isolated point iff it is sequential at every point.

**Theorem**

For distributive concrete domains $D$ and $E$, a function $f : D \rightarrow E$ is sequential iff it is sequential in the Kahn-Plotkin sense.
In other words...

- That is, $f$ is sequential iff it is continuous and for every state $x$ of $D$, either no cell is accessible from $x$, or for every cell $c'$ accessible from $f(x)$ there is a cell $c$ accessible from $x$ such that $c$ is filled in all states $y \supseteq x$ such that $c'$ is filled in $f(y)$.
The Pointwise Order

Stable

- Set inclusion on stable opens induces the pointwise order on stable functions.
- The union of a (set inclusion) directed family of stable opens is stable open.
- The pointwise lub of a (pointwise) directed family of stable functions is a stable function.

Sequential

- Set inclusion on sequential opens induces the pointwise order on sequential functions.
- The union of a (set inclusion) directed family of sequential opens is sequential open.
- The pointwise lub of a (pointwise) directed family of sequential functions is a sequential function.

Problem

Berry: application fails to be stable (or sequential) under the pointwise order, but is stable wrt the stable order.
The Stable Order

• The lobe inclusion order on stable opens is given by: $p_1 \subseteq p_2$ iff $\text{lobes}(p_1) \subseteq \text{lobes}(p_2)$.

• This induces the stable order on stable functions, defined by: $f \sqsubseteq g$ iff for every $q \in \text{St}E$, $f^{-1}(q) \subseteq g^{-1}(q)$.

• We write $(D \rightarrow^\text{st} E, \sqsubseteq)$ for the stably-ordered stable function space.

• For any stable functions $f, g : D \rightarrow E$, the following are equivalent:

  1. $f \sqsubseteq g$.
  2. $f \leq g$ and $f(x) = g(x) \land f(y)$ for every $x \leq y$.
  3. $f \leq g$ and $f(x) \land g(y) = g(x) \land f(y)$ for every $x \uparrow y$.
  4. $f \leq g$ and, for every $d \in D$ and $e \leq f(d)$,

     \[
     \{d' \leq d \mid e \leq f(d')\} = \{d' \leq d \mid e \leq g(d')\}.
     \]

• Thus our stable order generalizes Berry’s and Zhang’s definition of stable order, which were based on dI-domains.
Sequential Functions and Stable Order

• If $p$ is stable open, $p'$ is sequential open, and $p \sqsubseteq p'$, then $p$ is sequential open.

• If $f$ is stable, $g$ is sequential, and $f \sqsubseteq g$, then $f$ is sequential.

• The isolated elements of $(D \to^{\text{sq}} E, \sqsubseteq)$ are the isolated elements of $(D \to^{\text{st}} E, \sqsubseteq)$ that are also sequential.

• dI-domains are closed under the stably-ordered sequential function space.

• This improves on earlier results for KP-sequentiality:
  - KP-sequential functions only defined on concrete domains.
  - Concrete domains not closed under stably-ordered sequential function space.
Application is not Sequential

• $\text{app} : (\text{Bool}^3 \rightarrow \text{Bool}) \times \text{Bool}^3 \rightarrow \text{Bool}$

• Not sequential: $p = \text{app}^{-1}(\{\text{tt}\})$ has no index at $x = (\text{gf}_1, \bot, \bot, \bot)$.

  – Any cover $r$ of $x$ must have one of the forms:

    $$r = r_1 \times \text{up}(\bot) \times \text{up}(\bot) \times \text{up}(\bot)$$
    $$r = \text{up}(\text{gf}_1) \times r_2 \times \text{up}(\bot) \times \text{up}(\bot)$$
    $$r = \text{up}(\text{gf}_1) \times \text{up}(\bot) \times r_2 \times \text{up}(\bot)$$
    $$r = \text{up}(\text{gf}_1) \times \text{up}(\bot) \times \text{up}(\bot) \times r_2,$$

  where $r_1$ covers $\text{gf}_1$ and $r_2$ covers $\bot$ in $\text{Bool}$.

  – In first case, the element $(\text{gf}_1, \text{tt}, \text{ff}, \bot)$ of $p \cap \text{up}(x)$ is not in $r$.

  – In the other cases we can also find elements of $p \cap \text{up}(x)$ that are not contained in $r$.

  – Hence $I(x, p)$ is empty and $p$ is not sequential open.

• Application is not sequential since when we know that the function is at least $\text{gf}_1$ we can’t tell what needs to be evaluated further.

• Failure seems caused by assumption that functions are computed incrementally, as in Kahn-Plotkin.
FM-domains

• A Scott domain has the *finite meet* property (FM) iff the meet of every pair of isolated elements is isolated.

• An FM-domain is a Scott domain with property FM.

• dI-domains are FM-domains.

• The converse is not generally true, and FM-domains are a proper intermediate notion, between Scott domains and dI-domains.

• The following are equivalent in an FM-domain:

  (1) $p$ is sequential open.
  (2) $p$ is Scott open and is sequential at every finite point.

**Theorem**

• FM-domains are closed under product and under continuous function space, so FM-domains and continuous functions are a sub-ccc of the ccc of Scott domains and continuous functions.

• All domains occurring in the Scott continuous functions model of PCF are FM-domains.
Stable Functions on FM-domains

- FM-domains are closed under the pointwise-ordered stable function space.
- This improves on a result that the pointwise-ordered stable function space between dI-domains is a Scott domain (Berry).
- We restrict to FM-domains, because the poset of stable opens, ordered by inclusion, is not bounded complete for general Scott domains.
Example

- For example, consider the following Scott domain, where $\omega$ is the limit of an infinite ascending chain, and all other elements are isolated. The stable opens $\text{up}(\alpha)$ and $\text{up}(\beta)$ are upper-bounded under inclusion, but have no lub.

\[
\begin{array}{c}
\top \\
\alpha \\
\omega \\
\cdot \\
\cdot \\
2 \\
1 \\
0 \\
\beta
\end{array}
\]
Stable Completion in FM-domains

• For a Scott-open set $p$ in an FM-domain $D$, define

\[
\text{stc}(p) = \text{up} \{x_1 \land x_2 \mid x_1, x_2 \in p \land x_1 \uparrow x_2\}
\]
\[
\text{stc}^0(p) = p
\]
\[
\text{stc}^{n+1}(p) = \text{stc}(\text{stc}^n(p))
\]
\[
\text{stc}^*(p) = \bigcup \{\text{stc}^n(p) \mid n \geq 0\}.
\]

• For any Scott-open $p$,

– $\text{stc}(p)$ is Scott-open;

– $p \subseteq \text{stc}(p)$;

– $\text{stc}^*(p)$ is the least stable open that contains $p$.

• For a function $f : D \to E$ and $x \in D$, define

\[
\text{stc}(f)(x) = \bigvee \{f(z_1) \land f(z_2) \mid z_1, z_2 \in D_{\text{fin}} \land
\]
\[
z_1 \uparrow z_2 \land z_1 \land z_2 \leq x\}
\]
\[
\text{stc}^0(f) = f
\]
\[
\text{stc}^{n+1}(f) = \text{stc}(\text{stc}^n(f))
\]
\[
\text{stc}^*(f) = \bigvee \{\text{stc}^n(f) \mid n \geq 0\}.
\]

• If $f : D \to E$ is continuous and $f$ is dominated by a stable function $h$, then

– $\text{stc}(f)$ is a continuous function;

– $f \leq \text{stc}(f) \leq h$;

– $\text{stc}^*(f)$ is the least stable function that dominates $f$. 


Properties

• The lub of a bounded set $F$ of stable functions is $\text{stc}^*(\bigvee F)$, where $\bigvee F$ is the pointwise lub.

• If $f$ is isolated in $D \rightarrow^\text{ct} E$ then $\text{stc}(f)$ and $\text{stc}^*(f)$ are isolated, and $\text{stc}^*(f) = \text{stc}^n(f)$ for some $n$.

• The isolated elements of $D \rightarrow^\text{st} E$ are the isolated elements of $D \rightarrow^\text{ct} E$ that are stable.

• The pointwise meet of two stable functions is stable.

• For any FM-domains $D$ and $E$, $D \rightarrow^\text{st} E$ is an FM-domain.

Sequential Functions on FM-domains

• If $D$ is an FM-domain and $E$ is a flat domain then the sequential functions from $D$ to $E$, ordered pointwise, forms an FM-domain.
Further Research

• Our notion of sequentiality works well at first-order types.

• Would like to develop an extension to deal adequately with higher-order types. A suitable higher-order notion of sequentiality must not rely on the Kahn-Plotkin operational assumption.

• It seems essential that the syntactic type of a function be used in defining sequentiality, not just the domain structure.

• We are currently working out the details of a definition of sequentiality at type $\tau \to \tau'$ using the above definition at first-order types. This would make application sequential.

• We conjecture that there is a (non-trivial) sub-class of the FM-domains that is closed under the pointwise-ordered sequential function space.

• These developments may lead to a fully abstract sequential model...?