Algorithms in the Real World (15-853), Fall 08
Assignment #4 Due: Nov 13

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You can look up material on the web and books, but you cannot look up solutions to the given problems. You can work in groups, but must write up the answers individually.

**Problem 1: Absolute Inequalities (10pt)**

Let's say we augment linear programs to allow constraints to include absolute values (*e.g.* $|x_1| + 3|x_2| \leq b$). Can we solve all such problems in polynomial time? Show why or why not. (Assume $P \neq NP$.)

**Problem 2: Testing for Nesting of Polytopes (10pt)**

Let $S = \{x \in R^n : Ax \leq b\}$ and $T = \{x \in R^n : Bx \leq d\}$. Assuming that $S$ and $T$ are nonempty, describe a polynomial time algorithm (in $n$) for checking whether $S \subseteq T$.

**Problem 3: Integer & Linear Programming Formulations, Connectivity (20 points)**

We will start by considering the shortest (or cheapest) $s-t$ path problem (that is, find the shortest path from $s$ to $t$ under given positive edge lengths $l(e)$).
First, some notation. Given a directed graph $G = (V, E)$ and set $U \subseteq V$, we let $\partial_+ U := \{(u, v) \in E \mid u \in U,\ v \notin U\}$ be the set of edges leaving $U$. Let $\partial_- U := \{(v, u) \in E \mid u \in U,\ v \notin U\}$ be the set of edges entering $U$. Given variables $\{x_i | i \in I\}$ and a set $J \subseteq I$, we define $x(J) := \sum_{i \in J} x_i$. Similiarly for variables $y_i, f_i$ etc, we define $y(\cdot), f(\cdot)$ etc.

(a) Write down an IP that encodes the shortest $s-t$ path problem using only the following binary variables $\{x_e \mid e \in E\}$. Assume the input is a directed graph $G = (V, E)$ with strictly positive edge lengths $l : E \to \mathbb{N}$, and no self loops. Your IP should have the property that every optimal IP solution $x$ corresponds to a shortest length $s-t$ path $P$ whose edges are $\{e \mid x_e = 1\}$. Prove that your IP does in fact have this property, and that the IP is feasible iff an $s-t$ path exists. Make sure to **give a high level description of your IP: list the variables and what they are supposed to represent, give a one sentence description of what each (non-trivial) constraint represents, and give some brief intuition of why it should be correct.** [Hint: Fix any simple $s-t$ path $P$ and delete all edges not in $P$. Consider the in-degree and out-degree of each vertex in the remaining graph.]

(b) Here's an alternate, more complex IP formulation of the shortest $s-t$ path problem. [Why we should care about this formulation should become clear soon enough.] Our input is a directed graph $G = (V, E)$ with lengths $l : E \to \mathbb{N}$. ($G$ has no self loops). Below, our variables are $\{f_e | e \in E\}$ and $\{x_e | e \in E\}$. For edge $e = (u, v)$, we let $f_e$ and $f_{(u,v)}$ denote the same variable.

$$
\begin{array}{ll}
\min\ \sum_e l(e) x_e & \\
0 \leq f_e \leq x_e & \text{for all } e \in E \\
f(\partial_+\{s\}) - f(\partial_-\{s\}) \geq 1 & \\
f(\partial_-\{t\}) - f(\partial_+\{t\}) \geq 1 & \text{[technically this constraint is unnecessary]} \\
f(\partial_-\{v\}) = f(\partial_+\{v\}) & \text{for all } v \in V \setminus \{s, t\} \\
0 \leq x_e \leq 1 & \text{for all } e \in E \\
x_e \in \mathbb{Z} & \text{for all } e \in E
\end{array}
$$

Assume $l(e) > 0$ for each edge $e$. Show that the above IP formulation is correct. Specifically, prove that there is a one-to-one correspondence[1] between optimal solutions to the IP given above and shortest

[1] In general we don't need a 1-1 correspondence, only an efficient algorithm that converts any optimal IP solution to some optimal shortest path solution.

paths from $s$ to $t$ in $G$. Give an $O(|E|)$ time algorithm to compute a shortest path in $G$ given an optimal solution $(x, f)$ to the IP above. Hint: think of $f$ as a flow.

(c) Write down an IP formulation for the following generalized connectively (GCP) problem: Given a directed graph $G = (V, E)$, edge costs $c(e)$, and a connectively requirement function $r : V \times V \to \mathbb{N}$, find the minimum cost subgraph of $G$ such that for all $u, v \in V$, there are at least $r(u, v)$ edge disjoint paths from $u$ to $v$. Your formulation should have size polynomial in the size of $G$.

Prove your formulation is correct. First, **give a high level description of your IP: list the variables and what they are supposed to represent, give a one sentence description of what each (non-trivial) constraint represents, and give some brief intuition of why it should be correct.** Next, assume $c(e) > 0$ for each edge $e$. Next, show that if there exists a solution to the GCP instance, then your IP has a feasible solution. Finally, provide an efficient algorithm which, when given an optimal IP solution, constructs an optimal GCP solution. (Ideally, this algorithm should be very simple – no psuedocode, just a sentence or two). Prove its correctness.

**Problem 4: Branch and Bound (15 points)**

Solve the following problem using the implicit enumeration branch-and-bound technique (the 0-1 programming technique) describe in class. You should show the tree and mark on any leaf node why it was pruned.

$$\begin{array}{ll} \text{minimize} & 3x_1 + 2x_2 + 5x_3 + x_4 \\ \text{subject to} & -2x_1 + x_2 - x_3 - 2x_4 \leq -2 \\ & -x_1 - 5x_2 - 2x_3 + 3x_4 \leq -3 \\ & x_1, x_2, x_3, x_4 \text{ binary} \end{array}$$